Hamilton Jacobi Bellman equations in infinite dimensions with quadratic and superquadratic Hamiltonian

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Abstract

We consider Hamilton Jacobi Bellman equations in an inifinite dimensional Hilbert space, with quadratic (respectively superquadratic) hamiltonian and with continuous (respectively lipschitz continuous) final conditions. This allows to study stochastic optimal control problems for suitable controlled Ornstein Uhlenbeck process with unbounded control processes.

1 Introduction

In this paper we study semilinear Kolmogorov equations in an infinite dimensional Hilbert space H, in particular Hamilton Jacobi Bellman equations. More precisely, let us consider the following equation

$$\begin{cases} \frac{\partial v}{\partial t}(t,x) = -\mathcal{L}v(t,x) + \psi\left(\nabla v(t,x)\sqrt{Q}\right) + l(x), & t \in [0,T],\ x \in H \\ v(T,x) = \phi(x), \end{cases} \tag{1.1}$$

where $\mathcal{L}$ is the generator of the transition semigroup P_t related to the following Ornstein-Uhlenbeck process

$$\begin{cases} dX_t = AX_t dt + \sqrt{Q}dW_t, & t \in [0,T] \\ X_0 = x, \end{cases} \tag{1.2}$$

that is, at least formally,

$$(\mathcal{L}f)(x) = \frac{1}{2}(TrQ\nabla^2 f)(x) + \langle Ax, \nabla f(x)\rangle.$$

The aim of this paper is to to consider the case where ψ has quadratic or superquadratic growth, and to apply our results to suitable stochastic optimal control problems: to this aim we make some regularizing assumptions on the Ornstein Uhlenbeck transition semigroup. At first in equation (1.1) we consider the case of final condition ϕ lipschitz continuous: with this assumption we can solve the Kolmogorov equation with ψ quadratic and superquadratic. In the case of quadratic hamiltonian we can solve equation (1.1) also in the case of final condiotion ϕ only bounded and continuous. A similar result, with ψ quadratic and superquadratic and with final condition ϕ lipschitz continuous, is proved in [14] by means of a detailed study on weakly continuous semigroups, and making the assumption that the transition semigroup P_t is strong Feller. Here we include the degenerate case and we exploit the connection between PDEs and backward stochastic differential equations (BSDEs in the following).

Coming into more details, we assume that A and Q in equation (1.2) commute, so that, see [17], the transition semigroup P_t satisfies the following regularizing property: for every

$\phi \in C_b(H)$, for every $\xi \in H$, the function $P_t\phi$ is Gâteaux differentiable in the direction $\sqrt{Q}\xi$ and for $0 < t \leq T$,

$$\left| \nabla P_t[\phi](x)\sqrt{Q}\xi \right| \leq \frac{c}{t^{1/2}} \|\phi\|_\infty |\xi|. \tag{1.3}$$

In order to prove existence and uniqueness of a mild solution v of equation (1.1), we use the fact that v can be represented in terms of the solution of a suitable forward-backward system (FBSDE in the following):

$$\begin{cases} dX_\tau = AX_\tau d\tau + \sqrt{Q}dW_\tau, & \tau \in [t,T] \subset [0,T], \\ X_t = x, \\ dY_\tau = -\psi(Z_\tau)\,d\tau - l(X_\tau)\,d\tau + Z_\tau\,dW_\tau, \\ Y_T = \phi(X_T), \end{cases} \tag{1.4}$$

It is well known, see e.g. [21] for the finite dimensional case and [12] for the generalization to the infinite dimensional case, that $v(t,x) = Y_t^{t,x}$, so that estimates on v can be achieved by studying the BSDE

$$\begin{cases} dY_\tau = -\psi(Z_\tau)\,d\tau - l(X_\tau)\,d\tau + Z_\tau\,dW_\tau, & \tau \in [0,T], \\ Y_T = \phi(X_T), \end{cases} \tag{1.5}$$

Moreover, if ψ is quadratic, we can remove the lipschitz continuous assumption on ϕ and prove existence and uniqueness of a mild solution of equation (1.1) with ϕ continuous and bounded. The fundamental tool is an apriori estimate on Z, and the classical identification $Z_\tau^{t,x} = \nabla v(\tau, X_\tau^{t,x})\sqrt{Q}$: the fact that A and Q commute is crucial in proving this estimates on $\nabla v(t,x)\sqrt{Q}$ by means of backward stochastic differential equations. This estimate is obtained with techniques similar to the ones introduced in [2], and specialized in [22] in the quadratic case, to treat BSDEs with generator ψ with superquadratic growth and in a markovian framework. In [2] the Markov process X solves a finite dimensional stochastic differential equation, with constant diffusion coefficient and with drift not necessarily linear as in our case. In order to obtain an estimate on $Z_\tau^{t,x}$, some non degeneracy assumptions on the coefficients are made. In the present paper the process X is infinite dimensional and we need the coefficient A and $\sqrt{Q}$ commute. We note that not in [2] nor in [22] the estimate on Z is used in order to solve a PDE related.

We also cite the paper [3] where infinite dimensional Hamilton Jacobi Bellman equations with quadratic hamiltonian are solved: the generator $\mathcal{L}$ is related to a more general Markov process X then the one considered here in (1.2), and no assumptions on the coefficient are made, but only the case of final condition ϕ Gâteaux differentiable is treated.

We apply these results on equation (1.1) to a stochastic optimal control problem. Let us consider the controlled equation

$$\begin{cases} dX_\tau^u = \left[AX_\tau^u + \sqrt{Q}u_\tau \right] d\tau + \sqrt{Q}dW_\tau, & \tau \in [t,T] \\ X_t^u = x. \end{cases} \tag{1.6}$$

where the control u takes values in a closed subset K of H. Define the cost

$$J(t,x,u) = \mathbb{E}\int_t^T [l(X_s^u) + g(u_s)]ds + \mathbb{E}\phi(X_T^u).$$

for real functions l, ϕ and g on H. The control problem in strong formulation is to minimize this functional J over all admissible controls u. We notice that we treat a control problem with unbounded controls, and, in the case of superaquadratic hamiltonian, we require weak coercivity on the cost J. Indeed, we assume that, for $1 < q \leq 2$,

$$0 \leq g(u) \leq c(1+|u|)^q, \quad \text{and} \quad g(u) \geq C|u|^q \qquad \text{for every } u \in K : |u| \geq R$$

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so that the hamiltonian function

$$\psi(z) = \inf_{u \in K} \{g(u) + zu\}, \quad \forall z \in H,$$

has quadratic growth in z if $q = 2$, and superquadratic growth of order $p > 2$, the coniugate exponent of q, if $q < 2$.

Some example of operators A and Q commuting are listed in section 2, moreover this conditon is satisfied by a stochastic heat equation with coloured noise:

$$\begin{cases} \dfrac{\partial y}{\partial s}(s,\xi) = \Delta y(s,\xi) + \dfrac{\partial W^Q}{\partial s}(s,\xi), & s \in [t,T], \ \xi \in \mathcal{O}, \\ y(t,\xi) = x(\xi), \\ y(s,\xi) = 0, \quad \xi \in \partial\mathcal{O}. \end{cases} \tag{1.7}$$

Here $W^Q(s,\xi)$ is a Gaussian mean zero random field, such that the operator Q characterizes the correlation in the space variable. The bounded linear operator Q is diagonal with respect to the basis $\{e_k\}_{k\in\mathbb{N}}$ of eigenfactors of the Laplace operator with Dirichlet boundary conditions. Equation (1.7) can be reformulated in $H = L^2(\mathcal{O})$ as an Ornstein-Uhlenbeck process (1.2) with A and Q commuting.

The paper is organized as follows: in section 2 some results on the Ornstein-Uhlenbeck process are collected, in section 3 the Kolmogorov equation (1.1) is solved with ψ with superquadratic growth and ϕ lipschitz continuous, and these results are applied to optimal control, in section 4 the Kolmogorov equation (1.1) is solved with quadratic ψ and ϕ only continuous and again an application to control is briefly presented, finally in section 5 optimal control problems for a controlled heat equation are solved.

2 Preliminary results on the forward equation and its semigroup

We consider an Ornstein-Uhlenbeck process in a real and separable Hilbert space H, that is a Markov process X solution to equation

$$\begin{cases} dX_\tau = AX_\tau d\tau + BdW_\tau, & \tau \in [t,T] \\ X_t = x, \end{cases} \tag{2.1}$$

where A is the generator of a strongly continuous semigroup in H and B is a linear bounded operator from Ξ to H. We define a positive and symmetric operator

$$Q_\sigma = \int_0^\sigma e^{sA} BB^* e^{sA^*} ds.$$

Throughout the paper we assume the following.

Hypothesis 2.1   *1. The linear operator A is the generator of a strongly continuous semigroup $\left(e^{tA}, t \geq 0\right)$ in the Hilbert space H. It is well known that there exist $M > 0$ and $\omega \in \mathbb{R}$ such that $\left\|e^{tA}\right\|_{L(H,H)} \leq Me^{\omega t}$, for all $t \geq 0$. In the following, we always consider $M \geq 1$ and $\omega \geq 0$.*

2. B is a bounded linear operator from Ξ to H and Q_σ is of trace class for every $\sigma \geq 0$.

We notice that in some of the literature, in the case $\Xi = H$, in order to define the Ornstein-Uhlenbeck process, a bounded, symmetric and positive operator Q is considered, and in equation 2.1 B is replaced by $\sqrt{Q}$.

The process X is clearly time-homogeneous. It is well known that the Ornstein-Uhlenbeck semigroup can be represented as $P_{\tau-t} = P_{t,\tau}$, where

$$P_\tau [\phi] (x) := \int_H \phi (y) \mathcal{N} \left(e^{\tau A} x, Q_\tau\right) (dy),$$

and $\mathcal{N} \left(e^{\tau A} x, Q_\tau\right) (dy)$ denotes a Gaussian measure with mean $e^{\tau A} x$, and covariance operator Q_τ.

In the following we are mainly concerned with the case $\Xi = H$, so we can take $B = \sqrt{Q}$ and we assume that A and $\sqrt{Q}$ commute. This happens e.g. when $(e_n)_n$ is an orthonormal basis in H and A and Q have the spectral decomposition $Ae_n = -\alpha_n e_n$ and $Qe_n = \gamma_n e_n$ where α_n, $\gamma_n > 0$ and $\alpha_n \uparrow +\infty$. If the α_n are positive apart from a finite number the result is still true.

More in general let $\Xi = H$, $B = \sqrt{Q}$. Suppose that A is an unbounded, selfadjoint and negative defined operator, $A = A^* \leq 0$, $A : \mathcal{D}(A) \subset H \to H$, with inverse bounded. We consider the spectral representation of A

$$A = \int_{-\infty}^0 s dE(s).$$

and $Q = (-A)^\beta = \int_{-\infty}^0 (-s)^\beta dE(s)$, for some $\beta \in \mathbb{R}$. It turns out that

$$Q_t = \int_0^t e^{sA} (-A)^\beta e^{sA} ds = \frac{1}{2} \left(1 - e^{2At}\right) (-A)^{\beta-1}.$$

It is proved in [17] that in this case the Ornstein-Uhlenbeck semigroup satisfies some regularizing property. Namely the following hypothesis 2.2 holds true with $\alpha = 1/2$.

We briefly introduce the notion of $\sqrt{Q}$-differentiability, see e.g. [17]. We recall that for a continuous function $f : H \to \mathbb{R}$ the $\sqrt{Q}$-directional derivative $\nabla^{\sqrt{Q}}$ at a point $x \in H$ in direction $\xi \in H$ is defined as follows:

$$\nabla^{\sqrt{Q}} f(x;\xi) = \lim_{s \to 0} \frac{f\left(x + s\sqrt{Q}\xi\right) - f(x)}{s}, \ s \in \mathbb{R}.$$

A continuous function f is $\sqrt{Q}$-Gâteaux differentiable at a point $x \in H$ if f admits the $\sqrt{Q}$-directional derivative $\nabla^{\sqrt{Q}} f(x;\xi)$ in every directions $\xi \in H$ and there exists a functional, the $\sqrt{Q}$−gradient $\nabla^{\sqrt{Q}} f(x) \in \Xi^*$ such that $\nabla^{\sqrt{Q}} f(x;\xi) = \nabla^{\sqrt{Q}} f(x) \xi$.

Hypothesis 2.2 *For some $\alpha \in [0,1)$ and for every $\phi \in C_b(H)$, the function $P_\tau [\phi] (x)$ is $\sqrt{Q}$-differentiable with respect to x, for every $0 \leq t < \tau < T$. Moreover there exists a constant $c > 0$ such that for every $\phi \in C_b(H)$, for every $\xi \in \Xi$, and for $0 \leq t < \tau \leq T$,*

$$\left| \nabla^{\sqrt{Q}} P_\tau [\phi] (x) \xi \right| \leq \frac{c}{\tau^\alpha} \|\phi\|_\infty |\xi|. \tag{2.2}$$

In [17] hypothesis 2.2 is verified by relating $\sqrt{Q}$-differentiability to properties of the operators A and $\sqrt{Q}$. Namely if

$$\text{Im } e^{tA} \sqrt{Q} \subset \text{Im } Q_t^{1/2}. \tag{2.3}$$

and for some $0 \leq \alpha < 1$ and $c > 0$ the operator norm satisfies

$$\left\| Q_t^{-1/2} e^{tA} \sqrt{Q} \right\| \leq ct^{-\alpha}, \text{ for } 0 < t \leq T.$$

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then hypotheses 2.2 is satisfied.

We also notice that this can be proved with a procedure similar to the one use in [8] to prove the Bismut-Elworthy formula. Namely, see e.g [8], lemma 7.7.2, for every uniformly continuous function ϕ with bounded and uniformly continuous derivatives up to the second order,

$$\phi(X_\tau^{t,x}) = P_{t,\tau}\phi(x) + \int_t^\tau < \nabla P_{s,\tau}\phi(X_s^{t,x}), \sqrt{Q}dW_s > \qquad \mathbb{P} - a.s. \qquad (2.4)$$

By multiplying both sides of (2.4) by

$$\int_t^\tau < \nabla(X_s^{t,x})h, dW_s >$$

and by taking expectation one gets

$$\mathbb{E}\left(\phi(X_\tau^{t,x}) \int_t^\tau < \nabla(X_s^{t,x})h, dW_s >\right) = \mathbb{E}\left[\int_t^\tau < \sqrt{Q}\nabla P_{s,\tau}\phi(X_s^{t,x}), \nabla(X_s^{t,x})hds >\right] \qquad (2.5)$$

$$= E\left[\int_t^\tau < \nabla P_{s,\tau}\phi(X_s^{t,x}), \sqrt{Q}e^{(s-t)A})hds >\right]$$

$$= \int_t^\tau < \nabla^{\sqrt{Q}}\mathbb{E}P_{s,\tau}\phi(X_s^{t,x}), h > ds$$

$$= \int_t^\tau < \nabla^{\sqrt{Q}}P_{t,s}P_{s,\tau}\phi(x), h > ds$$

$$= (\tau - t) < \nabla^{\sqrt{Q}}P_{t,\tau}\phi(x), h >$$

By arguments similar to the ones used in [8], lemma 7.7.5, we get that for every bounded and continuous function ϕ, the function $P_\tau[\phi](x)$ is $\sqrt{Q}$-differentiable with respect to x, for every $0 \le t < \tau \le T$ and

$$\left|\nabla^{\sqrt{Q}}P_{t,\tau}[\phi](x)\xi\right| \le \frac{c}{(\tau - t)^{1/2}} \|\phi\|_\infty |\xi|.$$

An analogous result can be proved in the case of A and Q commuting and P transition semigroup of the perturbed Ornstein Uhlenbeck process

$$\begin{cases} dX_\tau = AX_\tau d\tau + \sqrt{Q}F(\tau, X_\tau) + \sqrt{Q}dW_\tau, & \tau \in [t, T] \\ X_t = x, \end{cases}$$

with F jointly continuous in t and x and lipschitz continuous in x uniformly with respect to t.

The model we have in mind consists of an heat equation. Namely let $\mathcal{O}$ be a bounded domain in $\mathbb{R}$. We denote by H the Hilbert space $L^2(\Omega)$ and by $\{e_k\}_{k \in \mathbb{N}}$ the complete orthonormal basis which diagonalizes Δ, endowed with Dirirchlet boundary conditions in $\mathcal{O}$. We consider the equation

$$\begin{cases} \dfrac{\partial y}{\partial s}(s,\xi) = \Delta y(s,\xi) + \dfrac{\partial W^Q}{\partial s}(s,\xi), & s \in [t,T],\ \xi \in \mathcal{O}, \\ y(t,\xi) = x(\xi), \\ y(s,\xi) = 0, \quad \xi \in \partial\mathcal{O}. \end{cases} \qquad (2.6)$$

Here $W^Q(s,\xi)$ is a Gaussian mean zero random field, such that the operator Q characterizes the correlation in the space variables. Namely the covariance of the noise is given by

$$\mathbb{E} < W^Q(s,\cdot), h >_H < W^Q(t,\cdot), k >_H = t \wedge s < Qh, k >_H.$$

In particular $W^Q(s,\xi)$ can be the Brownian sheet so that $\frac{\partial^2 W^Q}{\partial s \partial \xi}(s,\xi)$ in this case is the space-time white noise. More in general we think about a coloured noise and on Q we make the following assumptions:

Hypothesis 2.3 *The bounded linear operator $Q : H \to H$ is positive and diagonal with respect to the basis $\{e_k\}_{k\in\mathbb{N}}$, with eigenvalues $\{\lambda_k\}_{k\in\mathbb{N}}$.*

By previous assumptions it turns out that $\lambda_k \geq 0$. Note that $W^Q(s, \cdot)$ is formally defined by

$$W^Q(s, \cdot) = \sum_{k=1}^{n} Qe_k(\cdot)\beta_k(s)$$

where $\{\beta_k(s)\}_{k\in\mathbb{N}}$ is a sequence of mutually independent standard Brownian motions, all defined on the same stochastic basis $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$.

Equation (2.6) can be written in an abstract way in H as

$$\begin{cases} dX_\tau = AX_\tau d\tau + \sqrt{Q}dW_\tau, & \tau \in [t, T] \\ X_t = x, \end{cases} \tag{2.7}$$

where A is the Laplace operator with Dirirchlet boundary conditions, W is a cylindrical Wiener process in H and Q is its covariance operator.

3 The semilinear Kolmogorov equation: lipschitz continuous final condition

The aim of this section is to present exitence and uniqueness results for the solution of a semilinear Kolmogorov equation with the nonlinear term which is quadratic with respect to the $\sqrt{Q}$-derivative. The following arguments presented in this section work also in the case of ψ with superquadratic growth with respect to z.

More precisely, let $\mathcal{L}$ be the generator of the transition semigroup P_t, that is, at least formally,

$$(\mathcal{L}f)(x) = \frac{1}{2}(TrQ^*\nabla^2 f)(x) + \langle Ax, \nabla f(x)\rangle.$$

Let us consider the following equation

$$\begin{cases} \frac{\partial v}{\partial t}(t, x) = -\mathcal{L}v(t, x) + \psi\left(\nabla^{\sqrt{Q}}v(t, x)\right) + l(x), & t \in [0, T], \ x \in H \\ v(T, x) = \phi(x), \end{cases} \tag{3.1}$$

We introduce the notion of mild solution of the non linear Kolmogorov equation (3.1), see e.g. [12] and also [17] for the definition of mild solution when ψ depends only on $\nabla^{\sqrt{Q}}v$ and not on ∇v. Since $\mathcal{L}$ is (formally) the generator of P_t, the variation of constants formula for (3.1) is:

$$v(t, x) = P_{t,T}[\phi](x) - \int_t^T P_{t,s}\left[\psi(\nabla^{\sqrt{Q}}v(s, \cdot))\right](x)\, ds, - \int_t^T P_{t,s}[l](x)\, ds, \quad t \in [0, T], \ x \in H. \tag{3.2}$$

We use this formula to give the notion of mild solution for the non linear Kolmogorov equation (3.1); we have also to introduce some spaces of continuous functions, where we seek the solution of (3.1).

For $\alpha \geq 0$, let $C_\alpha([0, T] \times H)$ (denoted by $C([0, T] \times H)$ for $\alpha = 0$) be the linear space of continuous functions $f : [0, T) \times H \to \mathbb{R}$ such that

$$\sup_{t \in [0,T]} \sup_{x \in H} (T - t)^\alpha |f(t, x)| < +\infty.$$

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$C_\alpha\left([0,T]\times H\right)$ endowed with the norm

$$\|f\|_{C_\alpha} = \sup_{t\in[0,T]}\sup_{x\in H}(T-t)^\alpha\,|f(t,x)|,$$

is a Banach space.

We consider also the linear space $C_\alpha^s\left([0,T]\times H,H^*\right)$ (denoted by $C^s\left([0,T]\times H,H^*\right)$ for $\alpha=0$) of the mappings $L:[0,T)\times H\to H^*$ such that for every $\xi\in H$, $L\left(\cdot,\cdot\right)\xi\in C_\alpha\left([0,T]\times H\right)$. The space $C_\alpha^s\left([0,T]\times H,H^*\right)$ turns out to be a Banach space if it is endowed with the norm

$$\|L\|_{C_\alpha(H^*)} = \sup_{t\in[0,T]}\sup_{x\in H}(T-t)^\alpha\,\|L(t,x)\|_{H^*}.$$

In other words, $C_\alpha^s\left([0,T]\times H,H^*\right)$ can be identified with the space of the operators $L\left(H,C_\alpha\left([0,T]\times H\right)\right)$.

Definition 3.1 *Let $\alpha\in[0,1)$. We say that a function $v:[0,T]\times H\to\mathbb{R}$ is a mild solution of the non linear Kolmogorov equation (3.1) if the following are satisfied:*

1. *$v\in C_b\left([0,T]\times H\right)$;*

2. *$\nabla^{\sqrt{Q}}v\in C_\alpha^s\left([0,T]\times H,H^*\right)$: in particular this means that for every $t\in[0,T)$, $v\left(t,\cdot\right)$ is $\sqrt{Q}$-differentiable;*

3. *equality (3.2) holds.*

Existence and uniqueness of a mild solution of equation (3.1) is related to the study of the following forward-backward system: for given $t\in[0,T]$ and $x\in H$,

$$\begin{cases} dX_\tau = AX_\tau d\tau + \sqrt{Q}dW_\tau, & \tau\in[t,T]\subset[0,T], \\ X_t = x, \\ dY_\tau = -\psi(Z_\tau)\,d\tau - l(X_\tau)\,d\tau + Z_\tau\,dW_\tau, \\ Y_T = \phi(X_T), \end{cases} \tag{3.3}$$

and to the identification of $Z_t^{t,x} = \nabla_x Y_t^{t,x}\sqrt{Q}$. We extend the definition of X setting $X_s = x$ for $0\le s\le t$. The second equation in (3.3), namely

$$\begin{cases} dY_\tau = -\psi(Z_\tau)\,d\tau - l(X_\tau)\,d\tau + Z_\tau\,dW_\tau, & \tau\in[0,T], \\ Y_T = \phi(X_T), \end{cases} \tag{3.4}$$

is of backward type. Under suitable assumptions on the coefficients $\psi:H\to\mathbb{R}$, $l:H\to\mathbb{R}$ and $\phi:H\to\mathbb{R}$ we will look for a solution consisting of a pair of predictable processes, taking values in $\mathbb{R}\times H$, such that Y has continuous paths and

$$\|(Y,Z)\|_{\mathbb{K}_{cont}}^2 := \mathbb{E}\sup_{\tau\in[0,T]}|Y_\tau|^2 + \mathbb{E}\int_0^T|Z_\tau|^2\,d\tau < \infty,$$

see e.g. [20]. In the following we denote by $\mathbb{K}_{cont}\left([0,T]\right)$ the space of such processes.

The solution of (3.3) will be denoted by $(X_\tau, Y_\tau, Z_\tau)_{\tau\in[0,T]}$, or, to stress the dependence on the initial time t and on the initial datum x, by $(X_\tau^{t,x}, Y_\tau^{t,x}, Z_\tau^{t,x})_{\tau\in[0,T]}$. In the following we refer to [12] for the definition of the class $\mathcal{G}(H)$ of Gâteaux differentiable functions $f:H\to\mathbb{R}$ with strongly continuous derivative.

Hypothesis 3.1 *The maps $\psi : H \to \mathbb{R}$, $l : H \to \mathbb{R}$ and $\phi : H \to \mathbb{R}$ are Borel measurable, moreover ψ is Gâteaux differentiable, namely $\psi \in \mathcal{G}(H)$ () and for every $\xi_1, \xi_2 \in \Xi$, $|\psi(\xi_1) - \psi(\xi_2)| \leq (1 + |\xi_1|^{p-1} + |\xi_2|^{p-1})|\xi_1 - \xi_2|$, for $p \geq 2$. The maps l and ϕ belong to $C_b(H)$. Moreover from now on we assume that, unless modifying the value of l, $\psi(0) = 0$.*

We make differentiability assumptions on the coefficients of equation (3.4):

Hypothesis 3.2 *l and ϕ belong to $\mathcal{G}(H)$ and have bounded derivative.*

Assume that in hypothesis 3.1 $p = 2$. So by [15], under hypothesis 3.1 the BSDE (3.4) admit a unique solution and by [3], under the further assumption 3.2 setting $v(t,x) := Y_t^{t,x}$, it turns out that v is the unique mild solution of equation (3.1), and $\nabla^{\sqrt{Q}} v(t,x) = Z_t^{t,x}$. By assuming that A and Q commute, or, more in general, by assuming that hypothesis 2.2 holds true, also imposing a more restrive structure on the forward equation and on the backward equation, we will prove in section 4 an estimate on $Z_\tau^{t,x}$ depending on τ, t, T and $\|\phi\|_\infty$ but not on $\nabla\phi$. Thanks to this estimate we will prove that by setting

$$v(t,x) := Y_t^{t,x}, \tag{3.5}$$

it turns out that v is the unique mild solution of equation (3.1), and $\nabla^{\sqrt{Q}} v(t,x) = Z_t^{t,x}$ without assumption 3.2. We note that differentiability on l, thanks to the regularizing property of the semigroup, can be easily removed. So from now on we can consider the case of $l = 0$.

We go on in this section with the study of equation (3.1) with ψ superquadratic.

3.1 Local mild solution of the corresponding PDE

In this subsection we look for a local mild solution of equation (3.1), that is a mild solution in a small time interval. We work on the PDE following [14], but the same result can be achieved by working on the BSDE with a procedure similar to the one indicate in [2], section 4.1.

We start by proving existence of a local mild solution to equation (3.1), and then we look for a priori estimates for this local mild solution.

Theorem 3.3 *Assume that hypotheses 2.1, 2.2, 3.1, 3.2 hold true. Then equation (3.1) admits a unique local mild solution u on $[T - \delta, T]$, for some $0 < \delta < T$ according to definition 3.1.*

Proof. The first part of the proof is similar to the proof of theorem 2.9 in [17] and we partially omit it. Consider the product space $\Lambda : C([0,T] \times H) \times C^s([0,T] \times H, H^*)$ with the product norm. Let us also denote by Λ_{R_0} the closed ball of radius R_0, with respect to the product norm

$$\|(f,L)\|_{C([0,T]\times H) \times C^s([0,T]\times H, H^*)} = \|f\|_{C([0,T]\times H)} + \|L\|_{C^s([0,T]\times H, H^*)}.$$

Let us also define, for $(u,v) \in \Lambda_{R_0}$

$$\Gamma_1[v,w](t,x) = P_{t,T}[\phi](x) - \int_t^T P_{t,s}[\psi(w(s,\cdot))](x)\,ds - \int_t^T P_{t,s}[l](x)\,ds, \tag{3.6}$$

$$\Gamma_2[v,w](t,x) = \nabla^{\sqrt{Q}} P_{t,T}[\phi](x) - \int_t^T \nabla^{\sqrt{Q}} P_{t,s}[\psi(w(s,\cdot))](x)\,ds. - \int_t^T \nabla^{\sqrt{Q}} P_{t,s}[l](x)\,ds \tag{3.7}$$

Thanks to condition (2.2) and if δ is sufficiently small, Γ is well defined on Λ_{R_0} with values in itself. Indeed, let us take

$$R_0 = 2Me^{\omega T}(\|\phi\|_1 + T\|l\|_1)$$

It turns out that

$$
\begin{aligned}
\|\Gamma_1 \left[v, w\right], \Gamma_2 \left[v, w\right] \|_{C([0,T] \times H) \times C^s([0,T] \times H, \Xi^*)} \\
\leq (\|\phi\|_1 + T\|l\|_1 + (T-t)C_{R_0}R_0 + (T-t)^{1-\alpha}CC_{R_0})R_0 \\
\leq (\frac{1}{2} + (T-t)C_{R_0} + (T-t)^{1-\alpha}CC_{R_0})R_0,
\end{aligned}
$$

where C_{R_0} is the lipschitz constant of ψ if $|v|_{C^s([0,T] \times H, H^*)} \leq R_0$. For $t \in [T - \delta, T]$ and δ sufficiently small we get that $\frac{1}{2} + \delta C_{R_0} + \delta^{1-\alpha}CC_{R_0} < 1$, so that $\Gamma : \Lambda_{R_0} \to \Lambda_{R_0}$ Moreover, arguing as in [17], theorem 2.9, it is possible to show that Γ is a contraction in Λ_{R_0}, and so we are able to find a unique local mild solution to equation (3.1). $\qquad \square$

3.2 Equivalent representation of the mild solution

In this subsection we give an an alternative representation of the mild solution of equation (3.1). Let v be the local mild solution of equation (3.1), as stated in theorem 3.3. Let us define

$$
G(t, x) = \int_0^1 \nabla \psi(\lambda \nabla^{\sqrt{Q}} v(t, x)) d\lambda. \tag{3.8}
$$

We present in an informal way the object of this subsection. Equation (3.1) can be rewritten as

$$
\begin{cases} \frac{\partial v}{\partial t}(t, x) = -\mathcal{L}v(t, x) + \langle \sqrt{Q}G(t, x), \nabla v(t, x) \rangle + l(x), & t \in [0, T], \ x \in H \\ u(T, x) = \phi(x), \end{cases} \tag{3.9}
$$

Let us consider the Markov process $\Theta^{t,x}_\cdot$ solution to equation

$$
\begin{cases} d\Theta^{t,x}_\tau = A\Theta^{t,x}_\tau d\tau + \sqrt{Q}G(\tau, \Theta^{t,x}_\tau)d\tau + \sqrt{Q}dW_\tau, & \tau \in [t, T] \quad t \in [T - \delta, T] \\ \Theta_t = x, \end{cases} \tag{3.10}
$$

Let us denote by $R_{t,\tau}$ the transition semigroup of $\Theta^{t,x}_\cdot$ Following [14], since the operator $\mathcal{L}u(t, x) + \langle \sqrt{Q}G(t, x), \nabla u(t, x) \rangle$ is formally the generator of R, we want to prove that the mild solution of (3.1) for $t \in [T - \delta, t]$ can be represented as

$$
v(t, x) = R_{t,T} \left[\phi\right](x) - \int_t^T R_{t,s} \left[l\right](x) ds, \ \ t \in [T - \delta, T], \ x \in H. \tag{3.11}
$$

Representation (3.11) immediately gives an a priori estimate for the norm of v in $C([0, T], H)$.

In the following lemma we notice that equation (3.10) admits a unique mild solution in weak sense. First of all we state an existence and uniqueness result for equation (3.10).

Lemma 3.4 *Let hypothesis 2.1 and 3.1 on ψ hold true, let G be defined by (3.8), then equation (3.10) has a unique mild solution in weak sense, and this solution is unique in law.*

Proof. By the Girsanov theorem, since $|G(t, x)| \leq R_0$, there exists a probability measure $\tilde{\mathbb{P}}$, equivalent to the original one $\mathbb{P}$, such that

$$
\left\{ \tilde{W}_\tau = \int_0^\tau G(r, x)dr + W_\tau, \tau \geq 0 \right\}
$$

is a Brownian motion. In the probability space $(\Omega, \mathcal{F}, \tilde{\mathbb{P}})$, $\Theta^{t,x}$ is an Ornstein-Uhlenbeck process:

$$\begin{cases} d\Theta_\tau^{t,x} = A\Theta_\tau^{t,x}d\tau + \sqrt{Q}d\tilde{W}_\tau, & \tau \in [t,T] \\ \Theta_t^{t,x} = x, \end{cases}$$

and this guarantees existence and uniqueness in law of a weak solution to equation 3.10. This suffices to have the transition semigroup R well defined. $\qquad\qquad\qquad\qquad\qquad\square$

Next we want to prove that representation (3.11) holds true. A similar result is obtained in [14] by using the results in [5], [6] about Cauchy problems associated to weakly continuous semigroups, such as transition semigroup. Here we use the connection between PDEs and BSDEs. We notice that in [2] a BSDE in a Markovian framework with generator with superqadratic growth is solved also in the case of final datum continuous in x. With the following techniques we solve a semilinear Komogorov equation like (3.1), in the case of lipschitz continuous final datum. Notice that by asking some smoothing properties of the transition semigroups allows also in this case to identify in the corresponding BSDE $Z_t^{t,x}$ with $\nabla^{\sqrt{Q}}Y_t^{t,x}$. In this way our results can be applied to solve a related stochastic optimal control problem.

Proposition 3.5 *Assume that hypotheses 2.1, 2.2, 3.1, 3.2 hold true. Then the local mild solution v of (3.1) can be represented as in (3.11).*

Proof. Let u be the local mild solution of (3.1) and let us define

$$Y_\tau^{t,x} = v(\tau, X_\tau^{t,x}), \quad Z_\tau^{t,x} = \nabla^{\sqrt{Q}}v(\tau, X_\tau^{t,x}), \qquad (3.12)$$

where as usual for $0 \le \tau \le t$, $X_\tau^{t,x} = x$. It is well known that the pair of processes $(Y_\tau^{t,x}, Z_\tau^{t,x})_{0 \le \tau \le T}$ is solution of the BSDE (3.4). Moreover since v is a local mild solution of (3.1), $|Z_\tau^{t,x}| \le R_0$ $\mathbb{P}$-a.s.. Since by our assumption 3.1 ψ is locally lipschitz continuous it turns out that

$$f(\tau) := \begin{cases} \dfrac{\psi(Z_\tau^{t,x})}{|Z_\tau^{t,x}|^2}Z_\tau^{t,x} & \text{if } Z_\tau^{t,x} \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

is bounded. So, by using the techniques introduced in [4], by the Girsanov theorem there exists a probability measure $\tilde{\mathbb{P}}$, equivalent to the original one $\mathbb{P}$, such that

$$\left\{ \tilde{W}_\tau = -\int_0^\tau f(r)dr + W_\tau, \tau \ge 0 \right\}$$

is a Brownian motion. So in $(\Omega, \mathcal{F}, \tilde{\mathbb{P}})$ equation (3.4) can be rewritten as

$$\begin{cases} dY_\tau = -l(X_\tau)\, d\tau + Z_\tau\, d\tilde{W}_\tau, & \tau \in [0,T], \\ Y_T = \phi(X_T). \end{cases} \qquad (3.13)$$

We notice that in $(\Omega, \mathcal{F}, \tilde{\mathbb{P}})$ X is solution to

$$\begin{cases} dX_\tau = AX_\tau d\tau + \sqrt{Q}G(\tau, X_\tau)d\tau + \sqrt{Q}dW_\tau, & \tau \in [t,T] \\ X_t = x, \end{cases} \qquad (3.14)$$

and lemma 3.4 guarantees existence, uniqueness and regularity of the mild solution of this equation. Moreover with respect to the new probability measure $\tilde{\mathbb{P}}$ the transition semigroup of X coincides with $R_{t,T}$. In particular we notice that the law of (X, Y, Z) depends on the coefficients A, G, ψ, l, ϕ, on the initial condition x given at initial time t, but not on the probability space nor on the Wiener process. So in particular it turns out that again, since $Y_t^{t,x}$ is deterministic, it coincides with $u(t,x)$, mild solution of equation 3.9. So it turns out that mild representation 3.11 holds true. $\qquad\qquad\qquad\qquad\qquad\square$

Remark 3.6 *It is possible to prove that for the local mild solution u of equation (3.1) there exists $L > 0$ such that for every $x, y \in H$ and for every $t \in [T - \delta, T]$*

$$(T - t)^\alpha |\nabla^{\sqrt{Q}} v(t, x) - \nabla^{\sqrt{Q}} v(t, y)| \leq L|x - y| \tag{3.15}$$

This can be proved by showing that Γ defined in the proof of theorem 3.3 is a contraction in the product space $C([0, T] \times H) \times C^s_{1,\alpha}([0, T] \times H, H^*)$ where by $C^s_{1,\alpha}([0, T] \times H, H^*)$ we mean the space of the operators $L \in C^s_\alpha([0, T] \times H, H^*)$ such that $(T - t)^\alpha L(t, x)$ is lipschitz continuous in x uniformly with respect to t. We endow $C^s_{1,\alpha}([0, T] \times H, H^*)$ with the norm

$$\|L\|_{C_{1,\alpha}(H^*)} = \sup_{t \in [0,T]} \sup_{x \in H} \|L(t, x)\|_{H^*} + \sup_{t \in [0,T]} \sup_{x,y \in H} (T - t)^\alpha \|L(t, x) - L(t, y)\|_{H^*}.$$

and we endow $C([0, T] \times H) \times C^s_{1,\alpha}([0, T] \times H, H^*)$ with the product norm. Let us also denote by Λ_{R_0} the closed ball of radius R_0, with respect to the product norm

$$\|(f, L)\|_{C([0,T] \times H) \times C_{1,\alpha}([0,T] \times H, H^*)} = \|f\|_{C([0,T] \times H)} + \|L\|_{C_{1,\alpha}([0,T] \times H, H^*)}.$$

Let us also define Γ_1 and Γ_2 as in (3.6) and (3.7). We take R_0 and δ as in the proof of theorem 3.3. We have to prove that $\Gamma : \Lambda_{R_0} \to \Lambda_{R_0}$ and it is a contraction. For the firts point, in view of the results of theorem 3.3 we have to prove that $(T - t)^\alpha \Gamma_2[v, w](t, \cdot)$ is lipschitz continuous. For $\xi \in H$ we have

$$|(T - t)^\alpha (\Gamma_2[v, w](t, x) - \Gamma_2[v, w](t, y))|$$

$$|(T - t)^\alpha \nabla^{\sqrt{Q}} P_{t,T}[\phi](x)\xi - \nabla^{\sqrt{Q}} P_{t,T}[\phi](y)\xi| + (T - t)^\alpha \int_t^T |\nabla^{\sqrt{Q}} P_{t,s}[l](x)\xi - \nabla^{\sqrt{Q}} P_{t,s}[l](y)\xi| ds$$

$$+ (T - t)^\alpha \int_t^T |\nabla^{\sqrt{Q}} P_{t,s}[\psi(w(s, \cdot))](x)\xi - \nabla^{\sqrt{Q}} P_{t,s}[w(s, \cdot)](y)\xi| ds = I + II + III.$$

We start by estimating I, following e.g. [7] and [17] we get :

$$I = (T - t)^\alpha |\nabla^{\sqrt{Q}} \int_H (\phi(z + e^{(T-t)A}x) - \phi(z + e^{(T-t)A}y))\mathcal{N}(0, Q_{T-t})dz|$$

$$= (T - t)^\alpha |\int_H (\phi(z + e^{(T-t)A}x) - \phi(z + e^{(T-t)A}y))\langle Q_{T-t}^{-1/2} e^{(T-t)A}\sqrt{Q}\xi, Q_{T-t}^{-1/2}z\rangle \mathcal{N}(0, Q_{T-t})(dz)|$$

$$\leq (T - t)^\alpha M e^{\omega(T-t)} \|\phi\|_1 |x - y|_H \|Q_{T-t}^{-1/2} e^{(T-t)A}\sqrt{Q}\| |\xi| \leq M e^{\omega(T-t)} \|\phi\|_1 |x - y|_H |\xi$$

Arguing in a similar way it follows that

$$II \leq (T - t) M e^{\omega(T-t)} \|\phi\|_1$$

so that

$$I + II \leq R_0/2$$

For what concerns III we get

$$III = (T - t)^\alpha \int_t^T \int_H |\psi(w(s, z + e^{(s-t)A}x)) - \psi(w(s, z + e^{(s-t)A}y))|$$

$$\langle Q_{s-t}^{-1/2} e^{(s-t)A}\sqrt{Q}\xi, Q_{s-t}^{-1/2}z\rangle \mathcal{N}(0, Q_{s-t})(dz)ds$$

$$\leq (T - t)^\alpha R_0 C_{R_0} M e^{\omega(T-t)} |x - y|_H \int_t^T (T - s)^{-\alpha} \|Q_{s-t}^{-1/2} e^{(s-t)A}\sqrt{Q}\| |\xi| ds$$

$$\leq 2R_0 C_{R_0} (T - t)^{1-\alpha},$$

where we have used the fact that u is the local mild solution and ψ is locally lipschitz continuous. Also by the proof of theorem 3.3 it turns out that

$$\|\Gamma_1[v,w],\Gamma_2[v,w]\|_{C([0,T]\times H)\times C^{s,1,\alpha}([0,T]\times H,H^*)}$$
$$\leq (\|\phi\|_1 + T\|l\|_1 + (T-t)C_{R_0}R_0 + (T-t)^{1-\alpha}CC_{R_0})R_0$$
$$\leq (1 + (T-t)C_{R_0} + 3(T-t)^{1-\alpha}CC_{R_0})R_0,$$

Let $0 < \bar{\delta} \leq \delta$ be such that for $t \in [T - \bar{\delta}, T]$ we get that $1 + \bar{\delta}C_{R_0} + 3\bar{\delta}^{1-\alpha}CC_{R_0} < 1$, so that $\Gamma : \Lambda_{R_0} \to \Lambda_{R_0}$.

Moreover, it is possible to show that Γ is a contraction in Λ_{R_0}, and so we are able to find a unique local mild solution to equation (3.1) in $C([0,T]\times H)\times C^{s,1,\alpha}([0,T]\times H, H^*)$.

□

As a consequence equation (3.10) admits a unique mild solution, in classical (strong) sense.

Lemma 3.7 *Let hypothesis 2.1 and 3.1 on ψ hold true, let G be defined by (3.8), then equation (3.10) has a unique mild solution satisfying moreover, for every $x, y \in H$,*

$$|\Theta_\tau^{t,x} - \Theta_\tau^{t,y}| \leq C_T|x-y|$$

Proof. The proof is standard apart from the singularity of the Lipschitz constant of G in $T = t$: by theorem 3.3, the local mild solution u is Lipschitz continuous according to estimate 3.15, and so also G is. Indeed, for every $x, y \in H$,

$$(T-t)^\alpha|G(t,x) - G(t,y)| \leq L|x-y|$$

Existence, uniqueness and Lipschitz property of a mild solution of equation 3.10 follow as in [14], proposition 3.9. □

The aim of the next section is to find a priori estimates for the local mild solution of equation 3.1 by using reperesentation (3.11). We notice that, the transition semigroup $R_{t,T}$ is a perturbed Ornstein-Uhlenbeck transition semigroup, so we could try to investigate if it satisfies regularizing properties like the ones satisfied by the Ornstein-Uhlenbeck transition semigroup contained in 2.2. Anyway there are some difficulties related to the coefficient G in equation (3.10): G is not differentiable and blows up like $(T-t)^{-\alpha}$, so it is in general not square integrable since $0 < \alpha < 1$. In particular when A and Q commute, hypothesis 2.2 holds true with $\alpha = 1/2$. In the existing literature, see e.g. [10], [18] and references therein, regularizing properties for perturbed Ornstein-Uhlenbeck transition semigroup, such as the strong Feller property or property 2.2, are proved by means of "generalizations" of the Girsanov theorem and then by means of the Malliavin calculus, eventually with direct calculation of the Malliavin derivative. Here this cannot be done: since G is not square integrable, no immediate generalization of the Girsanov theorem can be applied. Also, G is not differentiable, so the existing theory cannot be directly used, even if in this direction generalizations seem less involved.

3.3 A priori estimates and global existence

In this section we investigate a priori estimates for the local mild solution of equation 3.1 that we have found in theorem 3.3. By proposition 3.5, the equivalent representation (3.11) for the mild solution v holds true and this immediately gives an a priori estimate for the supremum norm of v, namely

$$\|v\|_C \leq (\|\phi\|_0 + T\|l\|_0) \leq \frac{R_0}{2}. \tag{3.16}$$

Next, we look for a priori estimates for the norm $\|\cdot\|_{C(H^*)}$ of $\nabla^G v$.

In order to prove an a priori estimate for the norm $\|\cdot\|_{C(H^*)}$ of $\nabla^G v$, we exploit again the strict connection between PDEs and BSDEs. We start by the fact the if v is the local mild solution of (3.1), and $X^{t,x}$ is the Ornstein-Uhlenbeck process defined by (2.1), then $((v(\tau, X_\tau^{t,x}))_\tau, (\nabla v(\tau, X_\tau^{t,x}))_\tau)$ is solution to the the BSDE (3.4), that we rewrite for the reader convenience:

$$\begin{cases} dY_\tau^{t,x} = -\psi(Z_\tau^{t,x})\, d\tau - l(X_\tau^{t,x})\, d\tau + Z_\tau^{t,x}\, dW_\tau, & \tau \in [0,T], \\ Y_T^{t,x} = \phi(X_T^{t,x}), \end{cases} \tag{3.17}$$

For $\xi \in H$ let us define, if it exists, $F_\tau^{t,x} = \nabla^{\sqrt{Q}} Y_\tau^{t,x} \xi$, $V_\tau^{t,x} = \nabla^{\sqrt{Q}} Z_\tau^{t,x} \xi$. It turns out that such processes exist, $(F_\tau^{t,x}, V_\tau^{t,x}) \in \mathbb{K}_{cont}([0,T])$, and that they are solution to

$$\begin{cases} dF_\tau^{t,x} = -\nabla\psi(Z_\tau^{t,x}) V_\tau^{t,x}\, d\tau - \nabla l(X_\tau^{t,x}) e^{(\tau-t)A}\sqrt{Q}\xi\, d\tau + V_\tau^{t,x}\, dW_\tau, & \tau \in [0,T], \\ F_T^{t,x} = \nabla\phi(X_T^{t,x}) e^{(T-t)A}\sqrt{Q}\xi, \end{cases} \tag{3.18}$$

which is equation 3.17 differentiated in direction $\sqrt{Q}\xi$, since for $t \le \tau \le T$, $\nabla^{\sqrt{Q}} X_\tau^{t,x} \xi = e^{(\tau-t)A}\sqrt{Q}\xi$.

Proposition 3.8 *Assume that hypotheses 2.1, 2.2, 3.1 and 3.2 hold true. Then equation (3.18) admits a unique solution that is a pair of predictable processes, taking values in $\mathbb{R} \times H$, such that F has continuous paths and $F_t^{t,x}$ is bounded.*

Proof. We notice that again $Z_\tau^{t,x} = \nabla^{\sqrt{Q}} u(\tau, X_\tau^{t,x})$ where u is the local mild solution of equation 3.1. Since $\nabla\psi$ is locally lipschitz continuous, it turns out that

$$f_1(\tau) := \begin{cases} \dfrac{\nabla\psi(Z_\tau^{t,x})}{|Z_\tau^{t,x}|^2} Z_\tau^{t,x} & \text{if } Z_\tau^{t,x} \ne 0 \\[2mm] 0 & \text{otherwise}, \end{cases}$$

is bounded. Following again the method in [4] by the Girsanov theorem there exists a probability measure $\hat{\mathbb{P}}$, equivalent to the original one $\mathbb{P}$, such that

$$\left\{ \hat{W}_\tau = -\int_0^\tau f_1(r)dr + W_\tau, \quad \tau \ge 0 \right\}$$

is a Brownian motion. So in $(\Omega, \mathcal{F}, \hat{\mathbb{P}})$ equation (3.18) can be rewritten as

$$\begin{cases} dF_\tau^{t,x} = -\nabla l(X_\tau^{t,x}) e^{(\tau-t)A}\sqrt{Q}\xi\, d\tau + Z_\tau^{t,x}\, d\hat{W}_\tau, & \tau \in [0,T], \\ F_T^{t,x} = \nabla\phi(X_T^{t,x}) e^{(T-t)A}\sqrt{Q}\xi. \end{cases} \tag{3.19}$$

In this equation the generator $-\nabla l(X_\tau^{t,x}) e^{(\tau-t)A}\sqrt{Q}\xi$ is independent on F and V and it is bounded, so by classical theorems on BSDEs equation (3.19) admits a unique solution (F, V) such that F has continuous paths and

$$\|(F,V)\|_{\mathbb{K}_{cont}}^2 := \hat{\mathbb{E}} \sup_{\tau\in[0,T]} |F_\tau^{t,x}|^2 + \hat{\mathbb{E}} \int_0^T |V_\tau^{t,x}|^2\, d\tau < C,$$

see e.g. [20]. Notice that the constant C depends only on A, G, l, ψ on the initial condition x given at initial time t. In particular

$$|F_t^{t,x}| < Me^{\omega T}(\|\phi\|_1 + T\|l\|_1)$$

$\square$

We immediately deduce the following result:

Corollary 3.9 *Assume that hypotheses 2.1, 2.2 , 3.1, 3.2 hold true and let u be the local mild solution of equation (3.1), as stated in theorem 3.3.*

$$\|v\|_{C([0,T]\times H)} + \left\|\nabla^{\sqrt{Q}}v\right\|_{C([0,T]\times H,H^*)} \leq R_0. \tag{3.20}$$

Proof. The estimate for the norm $\|\cdot\|_C$ of u follows by (3.16), the estimate for the norm $\|\cdot\|_{C(H^*)}$ of $\nabla^{\sqrt{Q}}u$ follows by proposition 3.8. $\qquad\square$

We can state a result on existence and uniqueness of a mild solution u of equation (3.1), which immediately gives a unique mild solution of equation (3.4).

Theorem 3.10 *Assume that hypotheses 2.1, 2.2, 3.1 and 3.2 hold true. Then equation (3.1) admits a unique mild solution u according to definition 3.1. Let $X^{t,x}$ be solution of equation (2.1). The pair of processes $(Y_\tau^{t,x} = v(\tau, X_\tau^{t,x}), Z_\tau^{t,x} = \nabla^{\sqrt{Q}}v(\tau, X_\tau^{t,x}))_{\tau\in[0,T]}$ is the unique solution of the BSDE (3.4).*

Proof. The global existence of the mild solution v follows by the local existence (Theorem 3.3) and by the a priori estimates (Corollary 3.9). The connections between PDEs and BSDEs is classical in the literature also in the infinite dimensional case (see e.g. [12]) and the proof is complete. $\qquad\square$

The next step is to remove differentiability assumptions on l and ϕ. We start by assuming l bounded and continuous and ϕ bounded and lipschitz continuous.

Theorem 3.11 *Assume that hypotheses 2.1, 2.2, 3.1, hold true and that l is bounded and continous and ϕ is bounded and lipschitz continuous. Then equation (3.4) admits a unique solution, that is a pair of processes $(Y_\cdot^{t,x}, Z_\cdot^{t,x}) \in \mathbb{K}_{cont}([0,T])$. The function $v(t,x) = Y_t^{t,x}$ is the unique mild solution of equation (3.1) according to definition 3.1.*

Proof. We consider the inf-sup convolution of ϕ (see e.g. [16] and [8]) denoted by ϕ_n and defined by

$$\phi_n(x) = \sup_{z\in H}\left\{\inf_{y\in H}\left[\phi(y) + n\frac{|z-y|_H^2}{2}\right] - n|x-z|_H^2\right\}. \tag{3.21}$$

Similarly, let us define l_n the inf-sup convolution of l. It is well known that $\phi_n \in UC_b^{1,1}(H)$ and as n tends to $+\infty$, ϕ_n converges to ϕ uniformly. Moreover, see also [17], let us denote by L the Lipschitz constant of ϕ; then $|\nabla\phi| \leq L$.
Now let us denote by $(Y_\cdot^{n,t,x}, Z_\cdot^{n,t,x})$ the unique solution of the BSDE (3.4) with ϕ_n and l_n in the place of ϕ and l respectively. By standard results on BSDEs we know that as $n \to \infty$

$$\mathbb{E}\sup_{\tau\in[0,T]}\left|Y_\tau^{n,t,x} - Y_\tau^{t,x}\right|^2 + \mathbb{E}\int_0^T\left|Z_\tau^{n,t,x} - Z_\tau^{t,x}\right|^2 d\tau \to 0$$

and moreover

$$\|(Y^n, Z^n)\|_{\mathbb{K}_{cont}}^2 < C$$

where C is a constant independent on n. We need to prove some further regularity on Z let us denote by $(F_\cdot^{n,t,x}, V_\cdot^{n,t,x})$ the unique solution of the BSDE (3.18) with ϕ_n and l_n in the place of ϕ and l respectively. It turns out that

$$\mathbb{E}\sup_{\tau\in[0,T]}\left|F_\tau^{n,t,x}\right|^2 < C$$

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where C is a constant depending on L and on $\|\phi\|_\infty$, and independent on n. So the process $Z^{n,t,x}_\cdot$ is uniformly bounded in n, since for $\xi \in H$, $Z^{n,t,x}_\tau \xi = F^{n,t,x}_\tau$. We also get that

$$\mathbb{E} \sup_{\tau \in [0,T]} \left|Y^{t,x}_\tau\right|^2 + \mathbb{E} \sup_{\tau \in [0,T]} \left|Z^{t,x}_\tau\right|^2 < C.$$

By setting $v(t,x) = Y^{t,x}_t$, $\nabla^G v(t,x) = Z^{t,x}_t$, we have found a (unique) mild solution to (3.1). $\quad\square$

3.4   Application to control

We formulate the stochastic optimal control problem in the strong sense. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a given complete probability space with a filtration $(\mathcal{F}_\tau)_{\tau \geq 0}$ satisfying the usual conditions. $\{W(\tau), \tau \geq 0\}$ is a cylindrical Wiener process on H with respect to $(\mathcal{F}_\tau)_{\tau \geq 0}$. The control u is an $(\mathcal{F}_\tau)_\tau$-predictable process with values in a closed set K of a normed space U; in the following we will make further assumptions on the control processes. Let $R : U \to H$ and consider the controlled state equation

$$\begin{cases} dX^u_\tau = \left[AX^u_\tau + \sqrt{Q}R\left(u_\tau\right))\right] d\tau + \sqrt{Q}dW_\tau, & \tau \in [t,T] \\ X^u_t = x. \end{cases} \tag{3.22}$$

The solution of this equation will be denoted by $X^{u,t,x}_\tau$ or simply by X^u_τ. X is also called the state, u and $T > 0$, $t \in [0,T]$ are fixed. The special structure of equation (3.22) allows to study the optimal control problem related by means of BSDEs and (3.22) leads to a semilinear Hamilton Jacobi Bellman equation which is a special case of the Kolmogorov equation (3.1) we have studied in the previous sections. The occurrence of the operator $\sqrt{Q}$ in the control term is imposed by our techniques, on the contrary the presence of the operator R allows more generality.

Beside equation (3.22), define the cost

$$J(t,x,u) = \mathbb{E} \int_t^T [l\left(X^u_s\right) + g(u_s)]ds + \mathbb{E}\phi\left(X^u_T\right). \tag{3.23}$$

for real functions l, ϕ on H and g on U. The control problem in strong formulation is to minimize this functional J over all admissible controls u. We make the following assumptions on the cost J.

Hypothesis 3.12   *1. The function $\phi : H \to \mathbb{R}$ is lipschitz continuous and bounded;*

   2. $l : H \to \mathbb{R}$ is bounded and continuous;

   3. $g : U \to \mathbb{R}$ is mesurable; and for some $1 < q \leq 2$ there exists a constant $c > 0$ such that

$$0 \leq g(u) \leq c(1 + |u|^q) \tag{3.24}$$

   and there exist $R > 0$, $C > 0$ such that

$$g(u) \geq C|u|^q \qquad \text{for every } u \in K : |u| \geq R. \tag{3.25}$$

In the following we denote by $\mathcal{A}_d$ the set of admissible controls, that is the K-valued predictable processes such that

$$\mathbb{E} \int_0^T |u_t|^q dt < +\infty.$$

This summability requirement is justified by (3.25): a control process which is not q-summable would have infinite cost.

We denote by $J^*(t, x) = \inf_{u \in \mathcal{A}_d} J(t, x, u)$ the value function of the problem and, if it exists, by u^* the control realizing the infimum, which is called optimal control.

We make the following assumptions on R.

Hypothesis 3.13 $R : U \to H$ *is measurable and* $|R(u)| \leq C(1 + |u|)$ *for every* $u \in U$.

We have to show that equation (3.22) admits a unique mild solution, for every admissible control u.

Proposition 3.14 *Let u be an admissible control and assume that hypothesis 2.1 holds true. Then equation (3.22) amits a unique mild solution* $(X_\tau)_{\tau \in [t,T]}$ *such that* $\mathbb{E} \sup_{\tau \in [t,T]} |X_\tau|^q < \infty$.

Proof. The proof follows the proof of proposition 2.3 in [11], with suitable changes since in that paper the finite dimensional case in considered and the current cost g has quadratic growth with respect to u, that is in (3.25) $q = 2$.

In order to make an approximation procedure in (3.22) we introduce the sequence of stopping times

$$\tau_n = \inf \left\{ t \in [0, T] : \mathbb{E} \int_0^t |u_s|^q ds > n \right\}$$

with the ususal convention that $\tau_n = T$ if this set is empty. Following the approximation procedure used in the proof of proposition 2.3 in [11] we can prove that there exists a unique mild solution with the required q-integrability. $\qquad \square$

We define in a classical way the Hamiltonian function relative to the above problem:

$$\psi(z) = \inf_{u \in K} \{ g(u) + zR(u) \} \quad \forall z \in H. \tag{3.26}$$

We prove that the hamiltonian function just defined satisfies the polynomial growth conditions and the local lipschitzianity required in hypothesis 3.1.

Lemma 3.15 *The hamiltonian* $\psi : H \to \mathbb{R}$ *is Borel measurable, there exists a constant $C > 0$ such that*

$$-C(1 + |z|^p) \leq \psi(z) \leq g(u) + |z|(1 + |u|), \qquad \forall u \in K,$$

where p is the coniugate exponent of q. Moreover if the infimum in (3.26) is attained, it is attained in a ball of radius $C(1 + |z|^{p-1})$ that is

$$\psi(z) = \inf_{u \in K, |u| \leq C(1 + |z|^{p-1})} \{ g(u) + zR(u) \}, \quad z \in H,$$

and

$$\psi(z) < g(u) + zR(u) \quad \text{if } |u| > C(1 + |z|^{p-1}).$$

In particular it follows that ψ is locally lipschitz continuous, namely $\forall z_1, z_2 \in H$, for some $C > 0$,

$$|\psi(z_1) - \psi(z_2)| \leq C(1 + |z_1|^{p-1} + |z_2|^{p-1})|z_1 - z_2| \tag{3.27}$$

Proof. The measurability of ψ is straightforward. By assumption (3.25) we get

$$g(u) + zR(u) \geq C(|u|^q - |R|^q) - C_1|z|(1 + |u|) \tag{3.28}$$

where C and R are as in (3.25) and $C_1 > 0$, and by this it follows that

$$\psi(z) \geq \inf_{u \in U}\{g(u) + zR(u)\} \geq C(|u|^q - |R|^q) - C_1|z|(1 + |u|) \geq -C_2|z|^p - C_3$$

for suitable constants C_2 and C_3. Moreover

$$|\psi(z)| \leq g(u) + c|z|(1 + |u|).$$

Now we prove that the infimum is attained in the ball of radius $C(1 + |z|^{p-1})$. By (3.28),

$$g(u) + zR(u) \geq C|u|(|u|^{q-1} - \frac{C_1}{C}|z|) - C|R|^q - C_1|z|.$$

On the other hand, for some $u^0 \in K$:

$$g(u^0) + zR(u^0) \leq C_4(1 + |z|).$$

and so there exists a constant $\bar{C}$ such that if $|u| \geq \bar{C}(1 + |z|^{p-1})$ then

$$g(u) + zR(u) \geq g(u^0) + zR(u^0)$$

and the result follows from the continuity of g and R. Finally (3.27) now easily follows. $\qquad\square$

Remark 3.16 *We give an example of hamiltonian we can treat. Let $g(u) = |u|^q$, $1 < q \leq 2$. Then, if $R(u) = u$, the hamiltonian function turns out to be*

$$\psi(z) = \left(\left(\frac{1}{q}\right)^{1/(q-1)} - \left(\frac{1}{q}\right)^p\right)|z|^p$$

where $p \geq 2$ is the coniugate of q. With this example, for $p = 2$, the Hamilton Jacobi Bellman related can be solved with the ad hoc exponential transform, see e.g.[14]. Our theory cover also the case of hamiltonian functions not exactly equal to $|z|^2$.

We define

$$\Gamma(s, x, z) = \{u \in U : zR(u) + g(u) = \Psi(z)\};\tag{3.29}$$

if $\Gamma(z) \neq \emptyset$ for every $z \in H$, by [1], see Theorems 8.2.10 and 8.2.11, Γ admits a measurable selection, i.e. there exists a measurable function $\gamma : H \rightarrow U$ with $\gamma(z) \in \Gamma(z)$ for every $z \in \mathbb{R}$.

In the following theorem, in order to prove the so called fundamental relation, we have to make further assumptions concerning differentiability of the hamiltonian function ψ. These assumptions allow us to say that the Hamilton Jacobi Bellman equation relative to the above problem, which is given by equation (3.1), admits a unique mild solution by theorem 3.3. Moreover this solution can be represented by means of the solution of the BSDE (3.4), namely the solution is given by $v(t, x) = Y_t^{t,x}$. So, adequating to our context the techniques e.g. in [11], we can prove the fundamental relation for the optimal control.

Theorem 3.17 *Assume hypotheses 2.1, 2.2, 3.12 and 3.13 hold true, and assume that the hamiltonian function ψ satisfies Gâteaux differentiability assumptions stated in hypothesis 3.1. For every $t \in [0, T]$, $x \in H$ and for all admissible control u we have $J(t, x, u(\cdot)) \geq v(t, x)$, and the equality holds if and only if*

$$u_s \in \Gamma\left(\nabla^{\sqrt{Q}}v(s, X_s^{u,t,x})\right)$$

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Proof. For every admissible control $(u_t)_{t \in [0,T]}$, we define, for every $n \in \mathbb{N}$,

$$u_t^n = u_t 1_{|u_t| \leq n} + n 1_{|u_t| > n}.$$

Since $u \in L^q(\Omega \times [0,T])$, then $u^n \to u$ in $L^q(\Omega \times [0,T])$ and so $u^n \to u$ with respect to the measure $dt \times \mathbb{P}$. Since moreover the sequence $(u_n)_n$ is monotone, then the convergence holds for almost all $t \in [0,T]$ and $\mathbb{P}$-almost surely. Moreover it follows that

$$\int_0^T |R(u_s^n)|^2 ds \leq C \int_0^T (1 + |u_s^n|)^2 ds \leq C(1 + n^2).$$

Let us define

$$\rho_n = \exp \left(- \int_0^T R(u_s^n) dW_s - \frac{1}{2} \int_0^T |R(u_s^n)|^2 ds \right)$$

In the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ let X^{u^n} denote the solution of equation

$$\begin{cases} dX_\tau^{u^n} = \left[AX_\tau^{u^n} + \sqrt{Q} R\left(u_\tau^n\right) \right) \right] d\tau + \sqrt{Q} dW_\tau, & \tau \in [t, T] \\ X_t^{u^n} = x. \end{cases} \tag{3.30}$$

By the Girsanov theorem there exists a probability measure $\mathbb{P}^n$, equivalent to the original one $\mathbb{P}$, namely $\frac{d\mathbb{P}^n}{d\mathbb{P}} = \rho_n$, and such that $W_t^n := W_t + \int_0^t R(u_s^n) ds$ is a $\mathbb{P}^n$-Wiener process. In $(\Omega, \mathcal{F}, \mathbb{P}^n)$, X^{u^n} solves the following stochastic differential equation

$$\begin{cases} dX_\tau^{u^n} = AX_\tau^{u^n} + \sqrt{Q} dW_\tau^n, & \tau \in [t, T] \\ X_t^{u^n} = x. \end{cases} \tag{3.31}$$

Let us also denote by (Y^n, Z^n) the solution in $(\Omega, \mathcal{F}, \mathbb{P}^n)$ of the BSDE

$$\begin{cases} dY_\tau^n = -\psi(Z_\tau^n) \, d\tau - l(X_\tau^{u^n}) \, d\tau + Z_\tau^n \, dW_\tau^n, & \tau \in [0, T], \\ Y_T = \phi(X_T^{u^n}). \end{cases} \tag{3.32}$$

We notice that the law of (X^{u^n}, Y^n, Z^n) depends on the coefficients A, $\sqrt{Q}$, ψ, l and ϕ and not on the Wiener process nor on the probability space. So in particular $Y_t^{n,t,x} = v(t, x)$, where v has been defined in (3.5) and it is also the solution of the related Hamilton-Jacobi-Bellman equation. This fact will be crucial in order to study the convergence of (Y^n, Z^n) as $n \to +\infty$. In the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we denote by X^u the solution of equation (3.22), then

$$\mathbb{E} \sup_{t \in [0,T]} |X^{u^n} - X^u|^q \leq \mathbb{E} \sup_{t \in [0,T]} \left| \int_0^t e^{(t-s)A} \sqrt{Q}(u_s - n) 1_{|u_s| > n} ds \right|$$

$$\leq C(T, A, Q) \mathbb{E} \int_0^T |u_s - n|^q 1_{|u_s| > n} ds$$

and so $X^{u^n} \to X^u$ in $L^q(\Omega, C([0,T], H))$, with probability measure $\mathbb{P}$. By combining this fact with the previous arguments on the the $dt \times \mathbb{P}$- almost sure convergence of u^n we get that $X_t^{u^n} \to X_t^u$ $\mathbb{P}$-almost surely uniformly with respect to t.
Moreover $(Y_t^{n,t,x}, Z_t^{n,t,x}) = (v(t,x), \nabla^{\sqrt{Q}} v(t,x))$, and for all $\tau \in [t, T]$,

$$(Y_\tau^{n,t,x}, Z_t^{n,t,\tau}) = (v(\tau, X_\tau^{u^n,t,x}), \nabla^{\sqrt{Q}} v(\tau, X_\tau^{u^n,t,x}) \tag{3.33}$$

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and since we work with lipschitz continuous assumptions on the final cost ϕ and Gâteaux differentiability assumptions on ψ by theorem 3.3 we get that both v and $\nabla^{\sqrt{Q}}v$ are bounded and continuous. In the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, by using the pointwise convergence, of X^{u^n} to X^u, uniformly with respect to time, we deduce a pointwise convergence uniform in time of (Y^n, Z^n), passing through the identification (3.33) of Y^n and Z^n. Namely, in $(\Omega, \mathcal{F}, \mathbb{P})$, $(Y_\tau^{n,t,x}, Z_\tau^{n,t,x}) = (v(\tau, X_\tau^{u^n,t,x}), \nabla^{\sqrt{Q}}v(\tau, X_\tau^{u^n,t,x})) \to (v(\tau, X_\tau^{t,x}), \nabla^{\sqrt{Q}}v(\tau, X_\tau^{t,x}))$ $\mathbb{P}$-almost surely uniformly with respect to τ.

Now we are ready to prove the fundamental relation: we integrate the BSDE (3.32) in $[t, T]$: at first we write down the equation with rspect to the $\mathbb{P}^n$-Wiener process W^n and then we pass to the process W, which is a standard Wiener process in the original probability space $(\Omega, \mathcal{F}, \mathbb{P})$:

$$dY_t^n = \phi(X_T^{u^n}) + \int_t^T \psi(Z_s^n) \, ds + \int_t^T l(X_s^{u^n}) \, ds - \int_t^T Z_s^n \, dW_s^n$$

$$= \phi(X_T^{u^n}) + \int_t^T \psi(Z_s^n) \, ds + \int_t^T l(X_s^{u^n}) \, ds - \int_t^T R(u_s^n)ds - \int_t^T Z_s^n \, dW_s$$

We notice that by standard arguments since $Z^n \in L^2((\Omega, \mathcal{F}, \mathbb{P}^n) \times [0, T])$, then it also holds that $Z^n \in L^2((\Omega, \mathcal{F}, \mathbb{P}) \times [0, T])$. Now we integrate with respect to the original probability $\mathbb{P}$: by taking expectation in the previous integral equality

$$dY_t^n = \mathbb{E}\phi(X_T^{u^n}) + \mathbb{E}\int_t^T \psi(Z_s^n) \, ds + \mathbb{E}\int_t^T l(X^{u^n}s) \, ds - \mathbb{E}\int_t^T Z_s^n \, dW_s^n$$

$$= \mathbb{E}\phi(X_T^{u^n}) + \mathbb{E}\int_t^T \psi(Z_s^n) \, ds + \mathbb{E}\int_t^T l(X_s^{u^n}) \, ds - \mathbb{E}\int_t^T Z_s^n R(u_s^n)ds - \mathbb{E}\int_t^T Z_s^n \, dW_s$$

$$= \mathbb{E}\phi(X_T^{u^n}) + \mathbb{E}\int_t^T \psi(\nabla^{\sqrt{Q}}v(s, X_s^{u^n})) \, ds + \mathbb{E}\int_t^T l(X_s^{u^n}) \, ds - \mathbb{E}\int_t^T \nabla^{\sqrt{Q}}v(s, X_s^{u^n})R(u_s^n)ds$$

where in the last passage the stochastic integral has zero expectation, and we have identified Z_s^n with $\nabla^{\sqrt{Q}}v(s, X_s^{u^n})$. Next we also identify Y_t^n with $v(t, x)$ and then we let $n \to +\infty$:

$$v(t, x) = \mathbb{E}\phi(X_T^{u^n}) + \mathbb{E}\int_t^T \psi(\nabla^{\sqrt{Q}}v(s, X_s^{u^n})) \, ds + \mathbb{E}\int_t^T l(X_s^{u^n}) \, ds - \mathbb{E}\int_t^T \nabla^{\sqrt{Q}}v(s, X_s^{u^n})R(u_s^n)ds$$

$$\to \mathbb{E}\phi(X_T^u) + \mathbb{E}\int_t^T \psi(\nabla^{\sqrt{Q}}v(s, X_s^u)) \, ds + \mathbb{E}\int_t^T l(X_s^u) \, ds - \mathbb{E}\int_t^T \nabla^{\sqrt{Q}}v(s, X_s^u)R(u_s)ds.$$

By adding and subtracting $\mathbb{E}\int_t^T g(u_s)ds$ we get

$$J(t, x, u) = v(t, x) + \mathbb{E}\int_t^T \left[-\psi(\nabla^{\sqrt{Q}}v(s, X_s^u)) + \nabla^{\sqrt{Q}}v(s, X_s^u)R(u_s) + g(u_s)\right] ds, \qquad (3.34)$$

from which we deduce the desired conclusion. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

Under the assumptions of Theorem 3.17, let us define the so called optimal feedback law:

$$u(s, x) = \gamma\left(\nabla^{\sqrt{Q}}v(s, X_s^{u,t,x})\right), \qquad s \in [t, T], \ x \in H. \qquad (3.35)$$

Assume that the closed loop equation admits a solution $\{\overline{X}_s, \ s \in [t, T]\}$:

$$\overline{X}_s = e^{(s-t)A}x_0 + \int_t^s e^{(s-r)A}\sqrt{Q}dW_r + \int_t^s e^{(s-r)A}R(\gamma(\nabla^{\sqrt{Q}}v(s, \overline{X}_r)))dr, \qquad s \in [t, T]. \ (3.36)$$

Then the pair $(\overline{u} = u(s, \overline{X}_s), \overline{X})_{s \in [t,T]}$ is optimal for the coNntrol problem. We nevertheless notice that existence of a solution of the closed loop equation is not obvious, due to the lack of regularity of the feedback law u occurring in (3.36). This problem can be avoided by formulating the optimal control problem in the weak sense, following [9], see also [12] and [17].

By an *admissible control system* we mean

$$(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}, W, u(\cdot), X^u),$$

where W is an H-valued Wiener process, u is an admissible control and X^u solves the controlled equation (3.22). The control problem in weak formulation is to minimize the cost functional over all the admissible control systems.

Theorem 3.18 *Assume hypotheses 2.1, 2.2, 3.12 and 3.13 hold true, and assume that the hamiltonian function ψ satisfies Gâteaux differentiability assumptions stated in hypothesis 3.1. For every $t \in [0,T]$, $x \in H$ and for all admissible control systems we have $J(t,x,u(\cdot)) \geq v(t,x)$, and the equality holds if and only if*

$$u_s \in \Gamma\left(\nabla^{\sqrt{Q}} v(s, X_s^u)\right)$$

Moreover assume that the set-valued map Γ is non empty and let γ be its measurable selection.

$$u_\tau = \gamma(\nabla^{\sqrt{Q}} v(\tau, X_\tau^u)), \quad \mathbb{P}\text{-a.s. for a.a. } \tau \in [t,T]$$

is optimal.

Finally, the closed loop equation

$$\begin{cases} dX_\tau^u = \left[AX_\tau^u + \sqrt{Q}R\left(\gamma(\nabla^{\sqrt{Q}} v(\tau, X_\tau^u)))\right)\right] d\tau + \sqrt{Q}dW_\tau, \quad \tau \in [t,T] \\ X_t^u = x. \end{cases} \tag{3.37}$$

admits a weak solution $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}, W, X)$ which is unique in law and setting

$$u_\tau = \gamma\left(\nabla^{\sqrt{Q}} v(\tau, X_\tau)\right),$$

we obtain an optimal admissible control system (W, u, X).

Proof. The proof follows from the fundamental relation stated in theorem 3.17. The only difference here is the solvability of the closed loop equation in the weak sense: this is a standard application of the Girsanov theorem. Indeed, by lemma 3.15, the infimum in the hamiltonian is achieved in a ball of radius $C(1 + |z|^{p-1})$ and so for the optimal control u the following estimate holds true, $\mathbb{P}$-a.s. and for a.a. $\tau \in [t,T]$, $0 \leq t \leq T$:

$$|u_\tau| \leq C(1 + |Z_\tau^{t,x}|^{p-1}) = C(1 + |\nabla^{\sqrt{Q}} v(\tau, X_\tau^{t,x}|^{p-1}) \leq \bar{C}.$$

Thanks to this bound we can apply a Girsanov change of measure and the conclusion follows in a standard way. $\square$

4 The semilinear Komogorov equation in the quadratic case: continuous final condition

Let v be the mild solution of the semilinear Kolmogororv equation 3.1, with the nonlinear term which is quadratic with respect to the $\sqrt{Q}$-derivative and with final condition ϕ differentiable. The aim of this section is to present an estimate for the $\sqrt{Q}$-derivative of $u(t,x)$ depending on T, t, $\|\phi\|_\infty$ but not on the $\nabla\phi$. If X is finite dimensional, also for processes more general than the Onstein uhlenbeck process, this estimate has been obtained in [22] by imposing some conditions on the coefficient of the forward equation for X. With those conditions, by inverting ∇X and by techniques coming from BMO martingales, the estimate is proved. Also in [2] a similar estimate is proved with X finite dimensional, with a more restrictive structure than in [22] but with ψ also with superquadratic growth. Applications of this estimate to a related Kolmogorov equation are not exploited in [2] nor in [22].

Here we prove the estimate for $\nabla^{\sqrt{Q}}v$ when X is an infinite dimensional Ornstein Uhlenbeck process and A and Q commute. We apply this estimate to prove that there exists a mild solution of the semilinear Kolmogorov equation (3.1) with the nonlinear term which is quadratic with respect to the $\sqrt{Q}$-derivative. Also in the finite dimensional case, in the setting of [2] and of [22], solution of the related Kolmogorov equation can be achieved with our techniques.

Assume that A and $\sqrt{Q}$ commute and that v is the unique mild solution of equation (3.1). As already noticed, if ϕ and ψ are Gâteaux differentiable, then v is given by $v(t,x) = Y_t^{t,x}$ where $(Y_\tau^{t,x}, Z_\tau^{t,x})_{\tau \in [t,T]}$ solve the BSDE in the forward-backward system (3.3) with $l = 0$, that we rewrite here:

$$\begin{cases} dX_\tau = AX_\tau d\tau + \sqrt{Q}dW_\tau, & \tau \in [t,T] \subset [0,T], \\ X_t = x, \\ dY_\tau = -\psi(Z_\tau)\,d\tau + Z_\tau\,dW_\tau, \\ Y_T = \phi(X_T^{t,x}). \end{cases} \tag{4.1}$$

Moreover, by [15], see also [3], when ψ is quadratic in Z it turns out that $(\Phi(\tau) = \int_t^\tau Z_s dW_s)_{\tau \in [t,T]}$ is a BMO martingale and

$$\|\Phi\|_{BMO} = \sup_{\sigma \in [t,T]} \mathbb{E}\left[\int_\sigma^T Z_s^2 ds | \mathcal{F}_\sigma\right]^{1/2} < +\infty,$$

where the supremum is taken over all stopping times $\sigma \in [t,T]$ a.s. Moreover, as a consequence, the stochastic exponential martingale

$$\mathcal{E}(\Phi)_\tau = \mathcal{E}_\tau = \exp\left(\int_t^\tau Z_s dW_s - \frac{1}{2}\int_t^\tau Z_s^2 ds\right),$$

is uniformly integrable. We are ready to prove an estimate on Z, independent on $\nabla\phi$.

Theorem 4.1 *Let (Y, Z) be the solution of the BSDE in (4.1). Let A and $\sqrt{Q}$ satisfy hypothesis 2.1, and assume that A and $\sqrt{Q}$ commute. Let ϕ and ψ satisfy hypotheses 3.1 and 3.2 with $p = 2$. Then the folllowing estimate holds true:*

$$|Z_t^{t,x}| \leq C(T-t)^{-1/2}, \tag{4.2}$$

where C depends on t, T, A, $\|\phi\|_\infty$ and not on $\nabla\phi$.

Proof. Let us take the $\sqrt{Q}$-derivative in the BSDE in (4.1) in the direction $h \in H$. Let us denote $F_\tau^{t,x} = \nabla^{\sqrt{Q}}Y_\tau^{t,x}h$ and $V_\tau^{t,x} = \nabla^{\sqrt{Q}}Z_\tau^{t,x}h$. (F, V) solve the following BSDE

$$\begin{cases} dF_\tau^{t,x} = -\nabla\psi(Z_\tau^{t,x})V_\tau^{t,x}\,d\tau + V_\tau^{t,x}\,dW_\tau, \\ F_T^{t,x} = \nabla\phi(X_T^{t,x})e^{(T-t)A}\sqrt{Q}h, \end{cases} \tag{4.3}$$

Let us denote by $\mathbb{Q}$ the equivalent probability measure such that

$$dW_\tau^\mathbb{Q} := -\int_0^\tau \psi(Z_s^{t,x})ds + W_\tau$$

is a Wiener process. Notice that by our assumptions $\nabla\psi$ has linear growth with respect to Z, so $\left(\int_t^\tau \nabla\psi(Z_s^{t,x})dW_s\right)_{\tau\in[t,T]}$ is a BMO martingale and

$$\mathcal{E}_\tau = \exp\left(\int_t^\tau \psi(Z_s^{t,x})W_s - \frac{1}{2}\int_t^\tau |\psi(Z_s^{t,x})|^2 ds\right),$$

is a uniformly integrable martingale. Notice that $\dfrac{d\mathbb{P}}{d\mathbb{Q}} = \mathcal{E}_T$.

In $(\Omega, \mathcal{F}, \mathbb{Q})$, $(F^{t,x}, V^{t,x})$ solve the following BSDE:

$$\begin{cases} dF_\tau^{t,x} = V_\tau^{t,x}\, dW_\tau^\mathbb{Q}, \\ F_T^{t,x} = \nabla\phi(X_T^{t,x})e^{(T-t)A}h, \end{cases} \tag{4.4}$$

It turns out that $(F^{t,x})^2$ is a $\mathbb{Q}$-submartingale.
In $(\Omega, \mathcal{F}, \mathbb{Q})$, $(Y^{t,x}, Z^{t,x})$ solve the following BSDE:

$$\begin{cases} dY_\tau^{t,x} = -\psi(Z_\tau^{t,x})d\tau + \nabla\psi(Z_\tau^{t,x})Z_\tau^{t,x}\, d\tau + Z_\tau^{t,x}\, dW_\tau^\mathbb{Q}, \\ Y_T^{t,x} = \phi(X_T^{t,x}), \end{cases}$$

and by our assumptions the generator $-\psi(Z_\tau^{t,x}) + \nabla\psi(Z_\tau^{t,x})Z_\tau^{t,x}$ has quadratic growth with respect to Z, so again by [15] $(\int_t^\tau Z_s^{t,x}ds, \tau \in [t,T])$ is a BMO $\mathbb{Q}$-martingale, with BMO norm depending only on T, t, A and $\|\phi\|_\infty$.
Moreover let us denote again $Y_t^{t,x} = v(t,x)$. Since A and Q commute,

$$F_\tau^{t,x} :=< \nabla v(\tau, X_\tau^{t,x}), \sqrt{Q}h >=< \nabla_x v(\tau, X_\tau^{t,x}), e^{(\tau-t)A}\sqrt{Q}h >$$
$$=< \nabla_x v(\tau, X_\tau^{t,x}), \sqrt{Q}e^{(\tau-t)A}h >=< Z_\tau^{t,x}, e^{(\tau-t)A}h > .$$

With these facts we can prove the desired estimate, indeed, since $(F^{t,x})^2$ is a $\mathbb{Q}$-submartingale

$$\mathbb{E}^\mathbb{Q}\left[\int_t^T |F_s^{t,x}|^2 ds|\mathcal{F}_t\right] \geq |F_t^{t,x}|^2(T-t) = |Z_t^{t,x}h|^2(T-t).$$

Moreover

$$\mathbb{E}^\mathbb{Q}\left[\int_t^T |F_s^{t,x}|^2 ds|\mathcal{F}_t\right] = \mathbb{E}^\mathbb{Q}\left[\int_t^T |< Z_s^{t,x}, e^{(s-t)A}h > |^2 ds|\mathcal{F}_t\right] \leq \|\Phi\|_{BMO,\mathbb{Q}},$$

where $C_{t,T}$ depends only on T, t, A. So

$$|Z_t^{t,x}| \leq C(T-t)^{-1/2},$$

where C depends on t, T, A, $\|\phi\|_\infty$ and not on $\nabla\phi$. $\qquad\square$

Remark 4.2 *Notice that under the assumptions of theorem 4.1 $Z_t^{t,x} = \nabla^{\sqrt{Q}}v(t,x)$, where $v(t,x) = Y_t^{t,x}$ is the unique mild solution of equation 3.1. So estimate (4.2) gives a bound of $\nabla^{\sqrt{Q}}v$ in $C_\alpha^s([0,T] \times H, H^*)$, with $\alpha = 1/2$.*

Next we want to apply the result of theorem 4.1 to find a mild solution to equation 3.1.

Theorem 4.3 *Let A and $\sqrt{Q}$ satisfy hypothesis 2.1 and assume that A and $\sqrt{Q}$ commute. Let ϕ and ψ satisfy hypotheses 3.1. Then equation (3.1) admits a unique mild solution v according to definition 3.1.*

Proof. Let $\phi \in UC_b(H)$. We can define, see e.g. [8] and [16], the inf-sup convolutions ϕ_n of ϕ by setting, for $n \geq 1$,

$$\phi_n(x) = \inf_{y \in H} \left\{ \phi(y) + 2n \, |x - y|_H^2 \right\}, \tag{4.5}$$

It is well known that the inf-sup convolution ϕ_n of ϕ provides an approximation of ϕ in the norm of the uniform convergence, preseving the supremum norm, and for every n, ϕ_n is lipschitz continuous and Frechet differentiable, with derivative blowing up like n as $n \to +\infty$. So for every n equation

$$\begin{cases} \frac{\partial u}{\partial t}(t, x) = -\mathcal{L}u(t, x) + \psi\left(\nabla^{\sqrt{Q}} u(t, x)\right) + l(x), & t \in [0, T], \ x \in H \\ u(T, x) = \phi_n(x), \end{cases} \tag{4.6}$$

admits a unique mild solution v_n according to definition 3.1. Notice that $v_n(t, x) = Y^{n,t,x}$, where we denote by $(X^{t,x}, Y^{n,t,x}, Z^{n,t,x})$ the unique solution of a forward backward system like (3.3) with final condition ϕ replaced by ϕ_n. It is immediate to see from the backward equation that

$$|Y_t^{n,t,x} - Y_t^{k,t,x}| \leq C\|\phi_n - \phi_k\|_\infty$$

Indeed, let us consider the BSDE solved by $Y^{n,t,x} - Y^{k,t,x}$

$$\begin{cases} d(Y_\tau^{n,t,x} - Y_\tau^{k,t,x}) = -\left(\psi(Z_\tau^{n,t,x}) - \psi(Z_\tau^{k,t,x})\right) d\tau + \left(Z_\tau^{k,t,x} - Z_\tau^{k,t,x}\right) dW_\tau, \\ Y_T^{n,t,x} = \phi^n(X_T^{t,x}) - \phi^k(X_T^{t,x}). \end{cases} \tag{4.7}$$

Let us denote by $\mathbb{Q}^{n,k}$ the equivalent probability measure such that

$$dW_\tau^{\mathbb{Q}^{n,k}} := -\int_0^\tau \frac{\psi(Z_s^{n,t,x}) - \psi(Z_s^{k,t,x})}{|Z_s^{n,t,x}) - Z_s^{k,t,x}|} \chi_{\left\{Z_s^{n,t,x} - Z_s^{k,t,x} \neq 0\right\}} ds + W_\tau$$

is a Wiener process. Writing equation 4.7 in $(\Omega, \mathcal{F}, \mathbb{Q}^{n,k})$ we get the desired estimate. Notice that by our assumptions

$$\frac{\psi(Z_s^{n,t,x}) - \psi(Z_s^{k,t,x})}{|Z_s^{n,t,x} - Z_s^{k,t,x}|} \chi_{\left\{Z_s^{n,t,x} - Z_s^{k,t,x} \neq 0\right\}} \leq C(1 + |Z_s^{n,t,x}| + |Z_s^{k,t,x}|)$$

so,

$$\left(\int_t^\tau \frac{\psi(Z_s^{n,t,x}) - \psi(Z_s^{k,t,x})}{|Z_s^{n,t,x} - Z_s^{k,t,x}|} \chi_{\left\{Z_s^{n,t,x} - Z_s^{k,t,x} \neq 0\right\}} dW_s \right)_{\tau \in [t,T]}$$

is a BMO martingale and

$$\mathcal{E}_\tau^{n,k} = \exp\left(\int_t^\tau \frac{\psi(Z_s^{n,t,x}) - \psi(Z_s^{k,t,x})}{|Z_s^{n,t,x} - Z_s^{k,t,x}|} \chi_{\left\{Z_s^{n,t,x} - Z_s^{k,t,x} \neq 0\right\}} dW_s \right.$$
$$\left. -\frac{1}{2} \int_t^\tau |\frac{\psi(Z_s^{n,t,x})}{-} \psi(Z_s^{k,t,x})|Z_s^{n,t,x} - Z_s^{k,t,x}| \chi_{\left\{Z_s^{n,t,x} - Z_s^{k,t,x} \neq 0\right\}}|^2 ds \right),$$

is a uniformly integrable martingale, with $\frac{d\mathbb{P}}{d\mathbb{Q}^{n,k}} = \mathcal{E}_T^{n,k}$. By previous arguments we know that

$$|Y_t^{n,t,x} - Y_t^{k,t,x}| \leq C\|\phi\|_\infty$$

uniformly in n and k. Next we have to prove that

$$|Z_t^{n,t,x} - Z_t^{k,t,x}| \leq C\|\phi_n - \phi_k\|_\infty$$

We differentiate equation (4.7), rewritten in $(\Omega, \mathcal{F}, \mathbb{Q}^{n,k})$, in the direction $\sqrt{Q}h$, $h \in H$. We get

$$\begin{cases} d(F_\tau^{n,t,x} - F_\tau^{k,t,x}) = (V_\tau^{n,t,x} - V_\tau^{k,t,x}) \, dW_\tau, \\ F_T^{n,t,x} - F_T^{k,t,x} = (\nabla\phi^n(X_T^{t,x}) - \nabla\phi^n(X_T^{t,x}))e^{(T-t)A}\sqrt{Q}h, \end{cases} \tag{4.8}$$

So, in $(\Omega, \mathcal{F}, \mathbb{Q}^{n,k})$, $\left\{F_\tau^{n,t,x} - F_\tau^{k,t,x}, \tau \in [t,T]\right\}$ is a martingale:

$$\mathbb{E}^{\mathbb{Q}^{n,k}}\left[\int_t^T |F_s^{n,t,x} - F_s^{k,t,x}|^2 ds | \mathcal{F}_t\right] \geq |F_t^{n,t,x} - F_t^{k,t,x}|^2(T-t) = |(Z_t^{n,t,x} - Z_t^{k,t,x})h|^2(T-t).$$

Moreover

$$\mathbb{E}^{\mathbb{Q}^{n,k}}\left[\int_t^T |F_s^{n,t,x} - F_s^{k,t,x}|^2 ds | \mathcal{F}_t\right] = \mathbb{E}^{\mathbb{Q}^{n,k}}\left[\int_t^T |<Z_s^{n,t,x} - Z_s^{k,t,x}, e^{(s-t)A}h>|^2 ds | \mathcal{F}_t\right]$$

$$\leq C_{t,T}\mathbb{E}^{\mathbb{Q}^{n,k}}\left[\int_t^T |Z_s^{n,t,x} - Z_s^{k,t,x}|^2 ds | \mathcal{F}_t\right],$$

where $C_{t,T}$ is a bounded constant depending on t, T, A. Moreover in $(\Omega, \mathcal{F}, \mathbb{Q}^{n,k})$ by equation (4.7) we immediately get

$$\mathbb{E}^{\mathbb{Q}^{n,k}}\left[\int_t^T |Z_s^{n,t,x} - Z_s^{k,t,x}|^2 ds | \mathcal{F}_t\right] \leq \|\phi_n - \phi_k\|_\infty$$

and so

$$|Z_t^{n,t,x} - Z_t^{k,t,x}| \leq C_{t,T}\|\phi_n - \phi_k\|_\infty(T-t)^{-1/2}, \tag{4.9}$$

where $C_{t,T}$ is a bounded constant depending on t, T, A and not on $\nabla\phi$.

So we get that by setting $v^n(t,x) = Y_t^n(t,x)$, the solution of the Kolmogorov equation (4.6) $v^n(t,x)$ converges in $C([0,T] \times H)$ to $v(t,x)$, equal to $Y_t^{t,x}$. Moreover for every n, $Z_t^{n,t,x} = \nabla^{\sqrt{Q}}v^n(t,x)$, and by (4.9) $(\nabla^{\sqrt{Q}}v^n(t,x))_n$ is a Cauchy sequence in $C_{1/2}^s([0,T] \times H))$. So $\nabla^{\sqrt{Q}}v^n(t,x)$ converges in $C_{1/2}^s([0,T] \times H))$ to an element that we denote by $F(t,x)$. For every $n \geq 1$,

$$\frac{v^n(t, x + s\sqrt{Q}h) - v^n(t,x)}{s} \int_0^1 \nabla^{\sqrt{Q}}v^n(t, x + r\sqrt{Q}h)h \, dr.$$

As $n \to +\infty$ we get

$$\frac{v(t, x + s\sqrt{Q}h) - v(t,x)}{s} \int_0^1 F(t, x + r\sqrt{Q}h)h \, dr.$$

which gives $F(t,x)h = \nabla^{\sqrt{Q}}v(t,x)h$. It remains to see that for every $\tau \in [0,T]$, $\nabla^{\sqrt{Q}}v(\tau, X_\tau^{t,x})h = Z_\tau^{t,x}$, where $Z^{t,x}$ is the limit of $Z^{n,t,x}$ in $L^2(\Omega \times [0,T])$. It turns out that by previous calculations $\nabla^{\sqrt{Q}}v^n(\tau, X_\tau^{t,x}) \to \nabla^{\sqrt{Q}}v(\tau, X_\tau^{t,x})$ in $C_{1/2}^s([0,T] \times H))$. So $\nabla^{\sqrt{Q}}v(\tau, X_\tau^{t,x}) = Z_\tau^{t,x}$ $\mathbb{P}$ a.s. for a.a. $\tau \in [t,T]$. Since (Y,Z) solve the BSDE in (4.1), with $Y_t^{t,x} = v(t,x)$, by previous arguments we get $Z_t^{t,x} = \nabla^{\sqrt{Q}}v(t,x)$. By classical arguments we deduce that v solves equation (3.1). Moreover the solution is unique since the solution of the corresponding BSDE is unique. $\square$

4.1 A quadratic optimal control problem

We apply the result of the previous section to a control problem where the current cost has quadratic growth with respect to the control u and the final cost is only continuous. The fundamental relation and the existence of a solution of the closed loop equation cannot be achieved as in theorem 3.17 and 3.18 respectively, since this time $\mathbb{Z}_t^{t,x} = \nabla^{\sqrt{Q}} v(t,x)$ is not bounded.

Let X^u the solution of equation (3.22), and we have to minimize the cost functional (3.23) over all the admissible control u, where by admissible control we mean here an $(\mathcal{F}_t)_t$-predictable process, taking values in a closed subset K of a normed space U, such that

$$\mathbb{E} \int_0^T |u_s|^2 ds < +\infty.$$

This assumptions is natural this time since we assume here that the cost has quadratic growth at infinity, namely the cost must satisfy hypothesis 3.12 with $q = 2$. We define the hamiltonian function in a classical way as in 3.26. The hamiltonian satisfies the properties stated in lemma 3.15, in particular estimates (3.28) and (3.27) hold true with $p = 2$ and the infimum is achieved in a ball of radius $C(1 + |z|)$.

Theorem 4.4 *Assume that A and $\sqrt{Q}$ satisfy hypothesis 2.1 and commute. Let g and l satisfy point 2 and 3 of hypothesis 3.12 and let hypothesis 3.13 hold true; assume that the hamiltonian function ψ satisfies Gâteaux differentiability assumptions stated in hypothesis 3.1. For every $t \in [0,T]$, $x \in H$ and for all admissible control u we have $J(t,x,u(\cdot)) \geq v(t,x)$, and the equality holds if and only if*

$$u_s \in \Gamma\left(\nabla^{\sqrt{Q}} v\big(s, X_s^{u,t,x}\big)\right)$$

Proof. The proof follows from proposition 4.1 in [11], and by our assumption here we have also the identification $Z_\tau^{t,x} = \nabla^{\sqrt{Q}} v(\tau, X_\tau^{t,x})$.

With the assumptions of Theorem 4.4, we can define the so called optimal feedback law as we have done in 3.35. Since as we have already noticed in section 3.4 existence of a solution of the closed loop equation is not obvious, we formulate the optimal control problem in the weak sense, following [9], see also section 3.4.

Theorem 4.5 *Assumehat A and $\sqrt{Q}$ satisfy hypothesis 2.1 and commute. Let g and l satisfy point 2 and 3 of hypothesis 3.12 and let hypothesis 3.13 hold true; assume that the hamiltonian function ψ satisfies Gâteaux differentiability assumptions stated in hypothesis 3.1. For every $t \in [0,T]$, $x \in H$ and for all admissible control systems we have $J(t,x,u(\cdot)) \geq v(t,x)$, and the equality holds if and only if*

$$u_s \in \Gamma\left(\nabla^{\sqrt{Q}} v\big(s, X_s^u\big)\right)$$

Moreover assume that the set-valued map Γ and let γ be its measurable selection.

$$u_\tau = \gamma(\nabla^{\sqrt{Q}} v(\tau, X_\tau^u)), \quad \mathbb{P}\text{-a.s. for a.a. } \tau \in [t,T]$$

is optimal.

Finally, the closed loop equation 3.37 admits a weak solution $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}, W, X)$ which is unique in law and setting

$$u_\tau = \gamma\left(\nabla^{\sqrt{Q}} v(\tau, X_\tau)\right),$$

we obtain an optimal admissible control system (W, u, X).

Proof. The proof follows from the fundamental relation stated in theorem 4.4. For the solvability of the closed loop equation we refer to proposition 5.2 in [11]. $\square$

5 Optimal control problems for the heat equation

In this section we present some control problem related to a stochastic heat equation. As in section 2, when introducing equation (2.6), here $\mathcal{O}$ is a bounded domain in $\mathbb{R}$, $H = L^2(\mathcal{O})$ and $\{e_k\}_{k\in\mathbb{N}}$ is the complete orthonormal basis which diagonalizes Δ, endowed with Dirrchlet boundary conditions in $\mathcal{O}$. $Q : H \to H$ satisfies hypothesis 2.3, in particular $Qe_k = \lambda_k e_k$, $\lambda_k \geq 0$, $k \in \mathbb{N}$. We consider the following controlled heat equation

$$\begin{cases} \dfrac{\partial y}{\partial s}(s,\xi) = \Delta y(s,\xi) + + \displaystyle\sum_{k\in\mathbb{N}} \sqrt{\lambda_k}\left(\int_{\mathcal{O}} u_s(\eta)e_k(\eta)d\eta\right)e_k(\xi) + \dfrac{\partial W^Q}{\partial s}(s,\xi), \qquad s \in [t,T],\ \xi \in \mathcal{O}, \\ y(t,\xi) = x(\xi), \\ y(s,\xi) = 0, \quad \xi \in \partial\mathcal{O}. \end{cases}$$
(5.1)

where $u_s \in L^2(\mathcal{O})$ represents the control. In the following we denote by $\mathcal{A}_d$ the set of admissible controls, that is the real valued predictable processes such that

$$\mathbb{E}\int_0^T\left(\int_{\mathcal{O}}|u_t(\xi)|^2 d\xi\right)^{q/2}dt < +\infty.$$

and such that $u_t \in K$, where K is a closed subset of H, not necessarily coinciding with H. Here as in equation (2.6), $W^Q(s,\xi)$ is a Gaussian mean zero random field, such that the operator Q characterizes the correlation in the space variables. Our aim is to minimize over all admissible controls the cost functional

$$J(t,x(\xi),u) = \mathbb{E}\int_t^T\int_{\mathcal{O}}[\bar{l}(X_s^u(\xi)) + |u_s(\xi)|^q]d\xi ds + \mathbb{E}\int_{\mathcal{O}}\bar{\phi}(X_T^u(\xi))\ d\xi.$$
(5.2)

for real functions $\bar{\phi}$ and $\bar{l}$, and for $q \leq 2$.

We make the following assumptions on the cost J.

Hypothesis 5.1 *The function $\bar{\phi} : \mathbb{R} \to \mathbb{R}$ is lipschitz continuous and bounded; $\bar{l} : \mathbb{R} \to \mathbb{R}$ is bounded and continuous.*

Let us define, for $\xi \in H$

$$\phi(x) = \int_{\mathcal{O}}\bar{\phi}(x(\xi))\ d\xi, \quad l(x) = \int_{\mathcal{O}}\bar{l}(x(\xi))\ d\xi.$$
(5.3)

It turns out that if $\bar{l}$ and $\bar{\phi}$ satisfy hypothesis 5.1, then ϕ and l defined in (5.3) satisfy hypothesis 3.12. Moreover by defining $g(u) = \int_{\mathcal{O}}|u_s(\xi)|^q d\xi = |u|_{L^2(\mathcal{O})}$, then the hamiltonian function turns out to be $\psi(z) = (\frac{1}{q})^{\frac{1}{q-1}}\frac{1-q}{q}|z|^p$.

Moreover equation (5.1) can be written in an abstract way in H as

$$\begin{cases} dX_\tau = AX_\tau d\tau + \sqrt{Q}u_\tau + \sqrt{Q}dW_\tau, \quad \tau \in [t,T] \\ X_t = x, \end{cases}$$
(5.4)

where A is the Laplace operator with Dirrchlet boundary conditions, W is a cylindrical Wiener process in H and Q is its covariance operator. The control problem in its abstract formulation is to minimize over all admissible controls the cost functional

$$J(t,x,u) = \mathbb{E}\int_t^T[l(X_s^u) + |u_s|^q]ds + \mathbb{E}\phi(X_T^u).$$
(5.5)

By applying results in section 3.4, we get the following

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Theorem 5.2 *Let X^u be the solution of equation (5.1), let the cost be defined as in (5.2) and let 5.1 hold true. Moreover assume that the hamiltonian function ψ satisfies Gâteaux differentiability assumptions stated in hypothesis 3.1. For every $t \in [0,T]$, $x \in L^2(\mathcal{O})$ and for all admissible control u we have $J(t,x,u(\cdot)) \geq v(t,x)$, and the equality holds if and only if*

$$u_s \in \Gamma\left(\nabla^{\sqrt{Q}} v(s, X_s^{u,t,x})\right)$$

Moreover assume that the set-valued map Γ is nonempty and let γ

The closed loop equation admits a weak solution $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}, W, X)$ which is unique in law and setting

$$u_\tau = \gamma\left(\nabla^{\sqrt{Q}} v(\tau, X_\tau)\right),$$

we obtain an optimal admissible control system (W, u, X).

Proof. The proof follows from the abstract formulation of the problem, and by applying theorems 3.17 and 3.18. $\square$

Next we turn to an optimal control problem related to the controlled equation (5.4) with quadratic cost g and consequently quadratic hamiltonian function, and with final cost continuous. In this case, in order to perform the synthesis of the optimal control, we apply the results of section 4.1. Namely we consider equation (5.1). We have to minimize the cost functional

$$J(t, x(\xi, u)) = \mathbb{E}\int_t^T \int_{\mathcal{O}} [\bar{l}(X_s^u(\xi)) + \bar{g}(u_s(\xi))] d\xi ds + \mathbb{E}\int_{\mathcal{O}} \bar{\phi}(X_T^u(\xi))\, d\xi. \tag{5.6}$$

over all admissible controls, that is real valued predictable processes such that

$$\mathbb{E}\int_0^T \left(\int_{\mathcal{O}} |u_t(\xi)|^2 d\xi\right) dt < +\infty.$$

$\bar{\phi}$, $\bar{g}$ and $\bar{l}$ are real functions satisfying the following:

Hypothesis 5.3 *The function $\bar{\phi} : \mathbb{R} \to \mathbb{R}$ is continuous and bounded; $\bar{l} : \mathbb{R} \to \mathbb{R}$ is bounded and continuous; $g : \mathbb{R} \to \mathbb{R}$ is continuous and for every $u \in \mathbb{R}$*

$$0 \leq g(u) \leq c(1 + |u|^2)$$

and there exist $R > 0$, $C > 0$ such that

$$g(u) \geq C|u|^2 \qquad \text{for every } u \in K : |u| \geq R.$$

Equation (5.1) admits the abstract formulation given by (5.4) and the cost functional can be formulated in an abstract way as

$$J(t, x, u) = \mathbb{E}\int_t^T l(X_s^u) + g(u_s)ds + \mathbb{E}\phi(X_T^u).$$

with notation 5.3 and by setting moreover

$$g(u) = \int_{\mathcal{O}} \bar{g}(u(\xi))\, d\xi, \qquad .$$

It turns out that if $\bar{\phi}$, $\bar{l}$ and $\bar{g}$ satisfy hypothesis 5.3, then ϕ, l and g satisfy hypothesis 3.12 with $q = 2$.

Theorem 5.4 *Let X^u be the solution of equation (5.1), let the cost be defined as in 5.6 and let 5.3 hold true. Moreover assume that the hamiltonian function ψ satisfies Gâteaux differentiability assumptions stated in hypothesis 3.1. For every $t \in [0, T]$, $x \in L^2(\mathcal{O})$ and for all admissible control u we have $J(t, x, u(\cdot)) \geq v(t, x)$, and the equality holds if and only if*

$$u_s \in \Gamma\left(\nabla^{\sqrt{Q}} v(s, X_s^{u,t,x})\right)$$

Moreover assume that the set-valued map Γ is nonempty and let γ be its measurable selection. The closed loop equation admits a weak solution $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}, W, X)$ which is unique in law and setting

$$u_\tau = \gamma\left(\nabla^{\sqrt{Q}} v(\tau, X_\tau)\right),$$

we obtain an optimal admissible control system (W, u, X).

Proof. The proof follows from the abstract formulation of the problem, and by applying theorem 4.4. $\square$

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